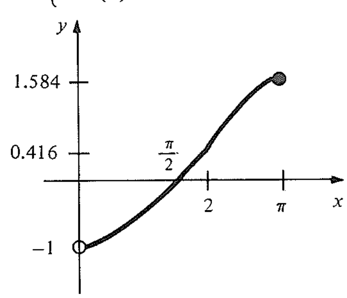
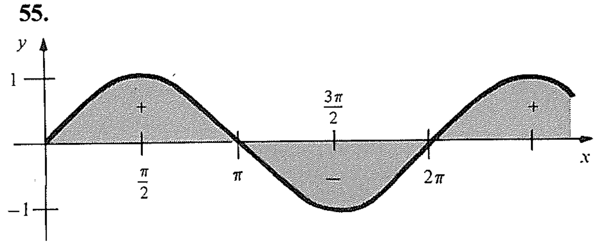
Chapter 7 Answers

7.1 Calculating Integrals

1. $x^3 + x^2 - 1/2x^2 + C$
3. $e^x + x^2 + C$
5. $-(\cos 2x)/2 + 3x^2/2 + C$
7. $-e^{-x} + 2\sin x + 5x^3/3 + C$
9. $1084/9$ **11.** $105/2$
13. $844/5$ **15.** $1/12$
17. 0 **19.** 6
21. $3\pi/4$ **23.** $\pi/12$
25. 1
27. $(e^6 - e^3)/3 + 3(2^{5/3} - 1)/5$
29. $\ln 5$ **31.** $4\ln 2 + 61/24$
33. 400 **35.** $116/15$
37. (b) $e^{(e^2)} - e + 3$
39. (a) 11
(b) -8
(c) Note that $\int_5^7 f(t)\,dt$ is negative
41. $-2t\sqrt{e^{t^2}} + \sin 5t^4$
43. 3
45. (a) 0
(b) $5/6$
(c) $\begin{cases} -\cos x & \text{if } 0 < x \leqslant 2 \\ \cos(2) - 2\cos x & \text{if } 2 < x \leqslant \pi \end{cases}$



47. $2 + \tan^{-1}2 - \dfrac{1}{2}\ln 5$
49. 16.4
51. $(1/2)(e^2 - 1)$
53. $16/3 - \pi$
55.



57. $\pi/4$
59. (a) Differentiate the right-hand side.
(b) Integrate both sides of the identity.
(c) $1/8$
61. Use the fact that $\tan^{-1}a$ and $\tan^{-1}b$ lie in the interval $(-\pi/2, \pi/2)$
63. 16,000,014 meters
65. (a) Evaluate the integral.
(b) $A = \$45{,}231.46$
67. (a) $R(t) = 2000e^{t/2} - 2000$, $C(t) = 1000t - t^2$
(b) \$57,279.90
69. $1 + \ln(2) - \ln(1 + e) \approx 0.380$

7.2 Integration by Substitution

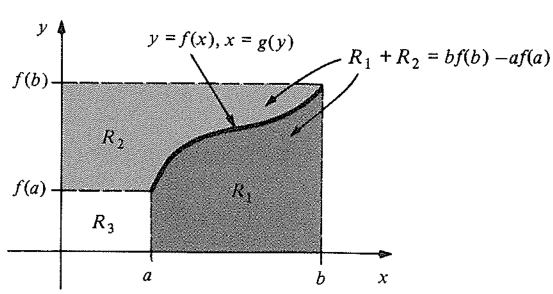
1. $\frac{2}{5}(x^2 + 4)^{5/2} + C$
3. $-1/4(y^8 + 4y - 1) + C$
5. $-1/2\tan^2\theta + C$
7. $\sin(x^2 + 2x)/2 + C$
9. $(x^4 + 2)^{1/2}/2 + C$
11. $-3(t^{4/3} + 1)^{-1/2}/2 + C$
13. $-\cos^4(r^2)/4 + C$
15. $\tan^{-1}(x^4)/4 + C$
17. $-\cos(\theta + 4) + C$
19. $(x^5 + x)^{101}/101 + C$
21. $\sqrt{t^2 + 2t + 3} + C$
23. $(t^2 + 1)^{3/2}/3 + C$
25. $\sin\theta - \sin^3\theta/3 + C$
27. $\ln|\ln x| + C$
29. $2\sin^{-1}(x/2) + x\sqrt{4 - x^2}/2 + C$
31. $\ln(1 + \sin\theta) + C$
33. $-\cos(\ln t) + C$
35. $-3(3 + 1/x)^{4/3}/4 + C$
37. $(\sin^2 x)/2 + C$
39. m a non-negative integer and n an odd positive integer, or n a non-negative integer and m an odd positive integer.

7.3 Changing Variables in the Definite Integral

1. $2(3\sqrt{3} - 1)/3$ **3.** $(5\sqrt{5} - 1)/3$
5. $2[(25)^{9/4} - (9)^{9/4}]/9$ **7.** $1/7$
9. $(e - 1)/2$ **11.** $-1/3$
13. 0 **15.** 1
17. $\ln(\sqrt{2}\cos(\pi/8))$ **19.** $1/2$
21. $4 - \tan^{-1}(3) + \pi/4$
23. (a) $\pi/2$
(b) $\pi/4$
(c) $\pi/8$
25. The substitution is not helpful in evaluating the integral.
27. $(\sqrt{2}/2)[\tan^{-1}2\sqrt{2} - \tan^{-1}(\sqrt{2}/2)]$
29. $(1/\sqrt{3})\ln[(4 + 3\sqrt{2})/(1 + \sqrt{3})]$
31. Let $u = x - t$.
33. $(5\sqrt{2} - 2\sqrt{5})/10$
35. $(\pi/27)(145\sqrt{145} - 10\sqrt{10})$
37. (a) $1/3$
(b) Yes.

7.4 Integration By Parts

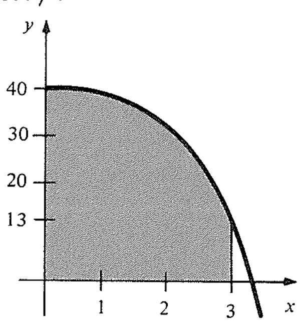
1. $(x+1)\sin x + \cos x + C$
3. $x \sin 5x/5 + \cos 5x/25 + C$
5. $(x^2-2)\sin x + 2x\cos x + C$
7. $(x+1)e^x + C$
9. $x\ln(10x) - x + C$
11. $(x^3/9)(3\ln x - 1) + C$
13. $e^{3s}(9s^2 - 6s + 2)/27 + C$
15. $(x^3-4)^{1/3}(x^3+12)/4 + C$
17. $t^2\sin t^2 + \cos t^2 + C$
19. $-(1/x)\sin(1/x) - \cos(1/x) + C$
21. $-[\ln(\cos x)]^2/2 + C$
23. $x\cos^{-1}(2x) - \sqrt{1-4x^2}/2 + C$
25. $y\sqrt{1/y - 1} - \tan^{-1}\sqrt{1/y - 1} + C$
27. $\sin^2 x/2 + C$
29. The integral becomes more complicated.
31. $(16+\pi)/5$ **33.** $3(3\ln 3 - 2)$
35. $\sqrt{2}[(\pi/4)^2 + 3\pi/4 - 2]/2 - 1$.
37. $\sqrt{3}/8 - \pi/24$
39. $e - 2$
41. $-(e^{2\pi} - e^{-2\pi})/4$
43. $\frac{3}{5}(2^{2/3}(2^{2/3}+1)^{5/2} - 2^{5/2} + \frac{2}{7}[2^{7/2} - (2^{2/3}+1)^{7/2}]) \approx 4.025$
45. $(\pi - 4)/8\sqrt{2} - 1/2$
47. $\int_0^1 \sqrt{2-x^2}\,dx - \int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = -\int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$ is $-1/8$ the area of a circle of radius $\sqrt{2}$ corrected by the area of a triangle (draw a graph).
49. $(-2\pi\cos 2\pi a)/a + (\sin 2\pi a)/a^2$. (This tends to zero as a tends to ∞. Neighboring oscillations tend to cancel one another.)
51. (b) $(5e^{3\pi/10} - 3)/34$
53. (a) Use integration by parts, writing $\cos^n x = \cos^{n-1}x \times \cos x$.
55. $2\pi^2$
57. (a) $Q = \int EC(\alpha^2/\omega + \omega)e^{-\alpha t}\sin(\omega t)\,dt$
(b) $Q(t) = EC\{1 - e^{\alpha t}[\cos(\omega t) + \alpha\sin(\omega t)/\omega]\}$
59.



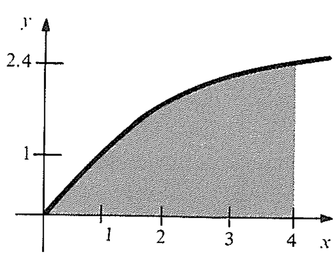
61. (a) $a_0 = 2$, all others are zero.
(b) $a_0 = 2\pi$, $b_n = -2/n$ if $n \neq 0$, all others are zero.
(c) $a_0 = 8\pi^2/3$, $a_n = 4/n^2$ if $n \neq 0$, $b_0 = 0$, $b_n = -4\pi/n$ if $n \neq 0$.
(d) $a_4 = b_2 = b_3 = 1$, all others are zero.

Review Exercises for Chapter 7

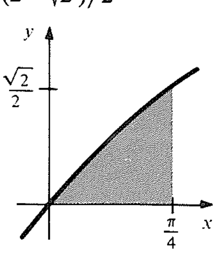
1. $x^2/2 - \cos x + C$
3. $x^4/4 + \sin x + C$
5. $e^x - x^3/3 - \ln|x| + \sin x + C$
7. $e^\theta + \theta^3/3 + C$ **9.** $-\cos(x^3)/3 + C$
11. $e^{(x^3)}/3 + C$ **13.** $(x+2)^6/6 + C$
15. $e^{4x^3}/12 + C$ **17.** $-\frac{1}{3}\cos^3 2x + C$
19. $x^2\tan^{-1}x/2 - x/2 + \tan^{-1}x/2 + C$
21. $\sin^{-1}(t/2) + t^3/3 + C$
23. $xe^{4x}/4 - e^{4x}/16 + C$
25. $x^2\sin x + 2x\cos x - 2\sin x + C$
27. $(e^{-x}\sin x - e^{-x}\cos x)/2 + C$
29. $x^3\ln 3x/3 - x^3/9 + C$
31. $(2/5)(x-2)(x+3)^{3/2} + C$
33. $x\sin 3x/3 + \cos 3x/9 + C$
35. $3x\sin 2x/2 + 3\cos 2x/4 + C$
37. $x^2e^{x^2}/2 - e^{x^2}/2 + C$
39. $x^2(\ln x)^2/2 - x^2(\ln x)/2 + x^2/4 + C$
41. $2e^{\sqrt{x}}(\sqrt{x} - 1) + C$
43. $\sin x\ln|\sin x| - \sin x + C$
45. $x\tan^{-1}x - \ln(1+x^2)/2 + C$
47. -1
49. $\pi/25$
51. $\sin(1) - \sin(1/2)$
53. $(\pi^2/32 + 1/2)\tan^{-1}(\pi/4) - \pi/8$
55. $(4\sqrt{2} - 2)/3 + (2\sqrt{2} - 2)a$
57. $3\sqrt{3}/5$
59. $399/4$



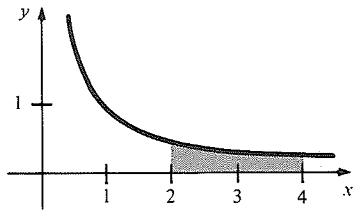
61. 6



63. $(2 - \sqrt{2})/2$



65. ln 2



67. $2/(n+1)$

69. 18.225

71. (a) 90008.46 liters
(b) 3000.28 liters/minute

73. $\frac{4}{3}[\sin(\pi x/2)\sin(\pi x)/\pi + \cos(\pi x/2)\cos(\pi x)/2\pi] + C$

75. $\sin^{-1}x - \sqrt{1-x^2} + C$

77. (a) $(\ln x)^2/2 + C$
(b) $(2/9)(-\sqrt{3}/3 + 1)$

79. $(x^{n+1}\ln x^{n+1} - x^{n+1})/(n+1)^2 + C$

81. (a) $(100/26)(\sin 5t/5 + \cos 5t + e^{-25t})$
(b) Substitute $t = 1.01$ in part (a).

83. (a) $m^2 + n^2 + mn + 2m + 2n + 1 = 0$. (b) The discriminant is negative. (c) Yes; for example $x^{-1/2}$ and $x^{(-3\pm\sqrt{5})/4}$.

85. $xe^{ax}[b\sin(bx) + a\cos(bx)]/(a^2+b^2) + e^{ax}[(b^2 - a^2)\cos(bx) - 2ab\sin(bx)]/(a^2+b^2)^2 + C$

Chapter 8 Answers

8.1 Oscillations

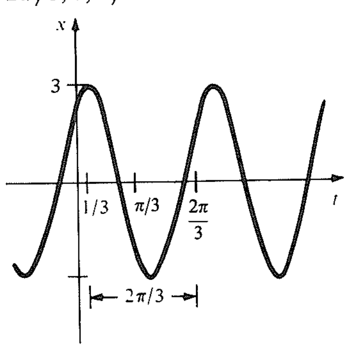
1. $\cos(3t) = \cos\left[3\left(t + \frac{2\pi}{3}\right)\right]$

3. $\cos(6t) + \sin(3t)$
$= \cos\left[6\left(t + \frac{2\pi}{3}\right)\right] + \sin\left[3\left(t + \frac{2\pi}{3}\right)\right]$

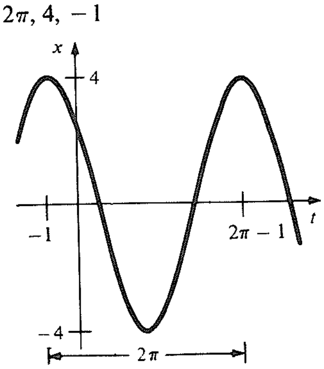
5. $\cos 3t - 2\sin 3t/3$

7. $-\frac{1}{6}\sqrt{3}\sin(2\sqrt{3}\,t)$

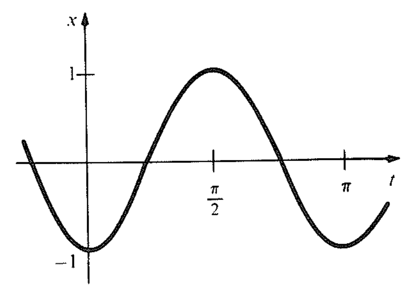
9. $2\pi/3$, 3, $1/3$



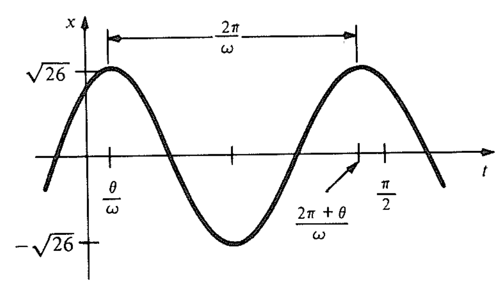
11. 2π, 4, -1



13. $-\cos 2t$



15. $\sqrt{26}\cos(5t - \tan^{-1}(1/5))$

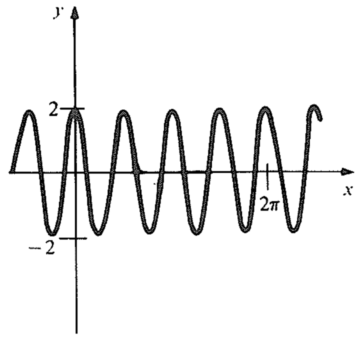


Phase shift $= \frac{\theta}{\omega} = \frac{\tan^{-1}(\frac{1}{5})}{5}$

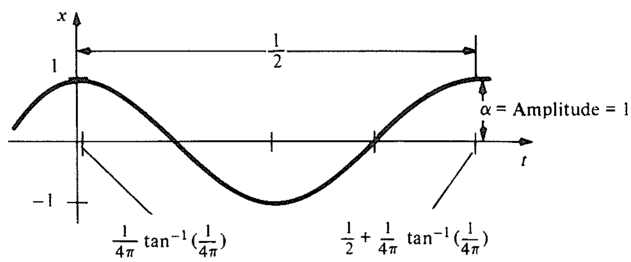
Period $= \frac{2\pi}{\omega} = \frac{2\pi}{5}$

17. $\cos 2t + (3/2)\sin 2t$

19. $2\cos 4x$



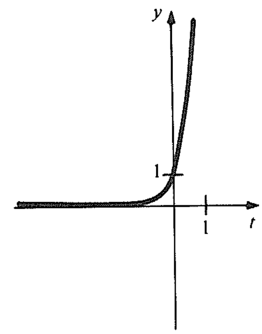
21. (a) $16\pi^2$
(b)



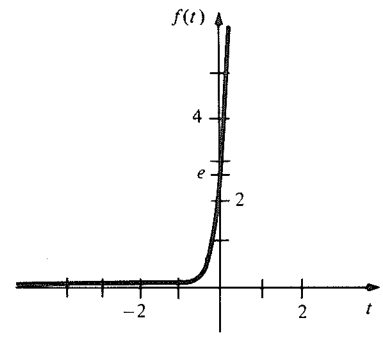
23. The frequency decreases by a factor of $\sqrt{3}$.
25. (a) $27(d^2x/dt^2) = -3x + 2x^3$
(b) $27(d^2x/dt^2) = -3x$
(c) 6π
27. (a) $x_0 = \dfrac{x_2 + x_1\sqrt[3]{k_2/k_1}}{1+\sqrt[3]{k_2/k_1}}$
(b) $f'(x_0) > 0$
29. There is no restriction on b.
31. Multiply (9) by $\omega \sin \omega t$ and (10) by $\cos \omega t$ and add.
33. (a) $V''(x_0) > 0$, so the second derivative test applies.
(b) Compute dE/dt using the sum and chain rules.
(c) Since E is constant, if it is initially small, the sum of $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2$ and $V(x)$ must remain small, so both dx/dt and $x - x_0$ remain small.

8.2 Growth and Decay

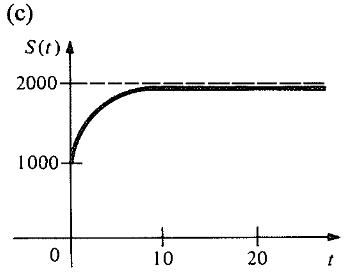
1. $dT/dt = -0.11(T-20)$
3. $dQ/dt = -(0.00028)Q$
5. $2e^{-3t}$
7. e^{3t}
9. $2e^{8t-8}$
11. $2e^{6-2s}$
13. 7.86 minutes
15. 2,476 years
17. e^{3t}



19. e^{8t+1}



21. Increasing
23. Decreasing
25. 33,000 years
27. 173,000 years
29. 1.5×10^9 years
31. 2,880 years
33. 49 minutes
35. 4.3 minutes
37. 18.5 years
39. The annual percentage rate is $100(e^{r/100} - 1) \approx 18.53\%$.
41. (a) $300\, e^{-0.3t}$
(b) 2000; 2000 books will eventually be sold.
(c)

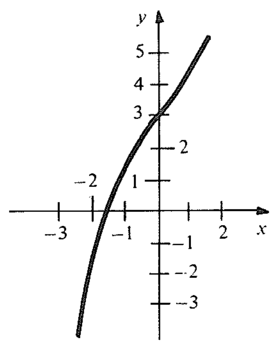


43. K is the distance the water must rise to fill the tank.
45. (a) Verify by differentiation.
(b) $a(t) = t(e^{-1/t} + 1 - e^{-1})$
47. $(2m/\delta)\ln 2$

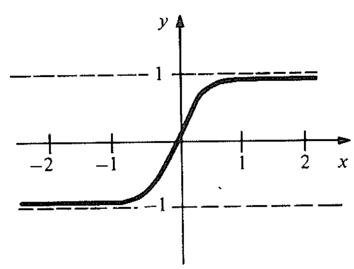
8.3 The Hyperbolic Functions

1. Divide (3) by $\cosh^2 t$.
3. Proceed as in Example 2.
5. $\dfrac{d}{dx}(\cosh x) = \dfrac{1}{2}\dfrac{d}{dx}(e^x + e^{-x}) = \dfrac{1}{2}(e^x - e^{-x}) = \sinh x$.
7. Use the reciprocal rule and Exercise 5.

9. $(3x^2+2x)\cosh(x^3+x^2+2)$
11. $\cosh x \sinh 5x + 5\cosh 5x \sinh x$
13. $-8\sin 8x \cosh(\cos 8x)$
15. $4\sinh x \cosh x$
17. $-3\operatorname{csch}^2 3x$
19. $(2\operatorname{sech}^2 2x)\exp(\tanh 2x)$
21. $[\sinh x(1+\tanh x) - \operatorname{sech} x]/(1+\tanh x)^2$
23. $(\sinh x)(\int[dx/(1+\tanh^2 x)]) +$
$\cosh x/(1+\tanh^2 x)$
25. $(\sinh 3t)/3$
27. $2\cosh\sqrt{3}\, t$
29. $\cosh 3t + (\sinh 3t)/3$
31. $2\cosh 6t$
33.



35.



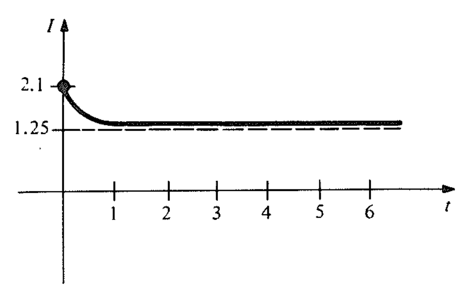
37. $(\sinh 3x)/3 + C$
39. $\ln|\sinh x| + C$
41. $(\sinh 2x)/4 - x/2 + C$
43. $e^{2x}/4 - x/2 + C$
45. $\cosh^3 x/3 + C$
47. $[y - \cosh(x+y)]/[\cosh(x+y) - x]$
49. $-3y\operatorname{sech}^2 3xy/(\cosh y + 3x\operatorname{sech}^2 3xy)$
51. (a) $x_0 = 1$
(b) $d^2x/dt^2 = 2(x-1)$
53. Use the definitions of $\sinh x$ and $\cosh x$. (Don't expand the nth power!)

8.4 The Inverse Hyperbolic Functions

1. $2x/\sqrt{x^4+4x^2+3}$
3. $(3-\sin x)\Big/\sqrt{(3x+\cos x)^2+1}$
5. $\tanh^{-1}(x^2-1) + 2/(2-x^2)$
7. $[(1+1/\sqrt{x^2-1})(\sinh^{-1}x + x) -$
$(x+\cosh^{-1}x)(1+1/\sqrt{x^2+1})]/(\sinh^{-1}x+x)^2$
9. $[\exp(1+\sinh^{-1}x)]/\sqrt{x^2+1}$
11. $-3\sin 3x/\sqrt{\cos^2 3x + 1}$
13. 0.55
15. 1.87
17. Let $y = \cosh^{-1}x$, so $x = \frac{1}{2}(e^y + e^{-y})$. Multiply by $2e^y$, solve the resulting quadratic equation for e^y and take logs.
19. Let $y = \operatorname{sech}^{-1}x$ so $x = 2/(e^y+e^{-y})$. Invert and proceed as in Exercise 17.
21. $$\frac{d}{dx}\tanh^{-1}x = \frac{1}{\dfrac{d}{dy}\tanh y} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}.$$
23. $$\frac{d}{dx}\operatorname{sech}^{-1}x = \frac{1}{\dfrac{d}{dy}\operatorname{sech} y} = \frac{1}{-\operatorname{sech} y \tanh y} = \frac{-1}{x\sqrt{1-\operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1-x^2}}$$
25. Differentiate the right hand side.
27. Differentiate the right hand side.
29. $(1/4)\ln|(1+2x)/(1-2x)| + C$
31. $(1/2)\ln(2x+\sqrt{4x^2+1}) + C$
33. $\ln(\sin x + \sqrt{\sin^2 x + 1}) + C$
35. $(1/2)\ln|(1+e^x)/(1-e^x)| + C$
37. No

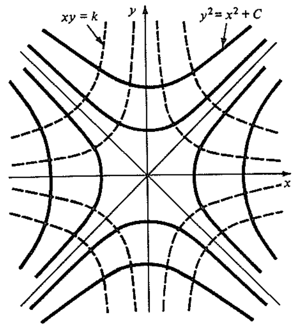
8.5 Separable Differential Equations

1. $y = \sin x + 1$
3. $y = \exp(x^2 - 2x + 1) - 1$
5. $y = -2x$
7. $e^y(y-1) = (1/2)\ln(x^2+1)$
9. $y = 2x + 1$
11. $y = \exp(-\sin x) + 1$
13.

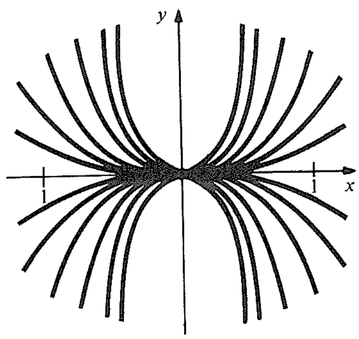


15. (a) $Q = EC(1 - \exp(-t/RC))$
(b) $t = RC\ln(100)$
17. Verify that the equations hold with $dx/dt = 0$ and $dy/dt = 0$.
19. $P = P_0 A\exp(P_0kt)/[1 + A\exp(P_0kt)]$
21. As T_0 increases, $\cosh\left(\dfrac{mgx}{T_0}\right) \to 1$, so $y \to h$, which represents a straight cable.

23. (a) $y' = -y/x$
(b) $y' = x/y$; $y^2 = x^2 + C$.

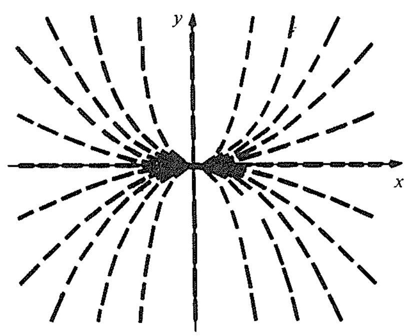


25. (a)



(b) $y' = 3cx^2$
(c) $y' = -1/3cx^2$; $y = 1/3cx + C$

27. (a)

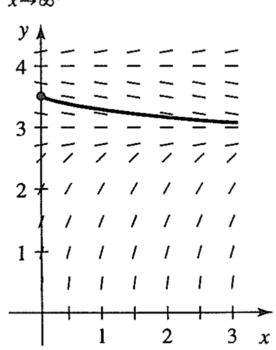


(b) $y = kx^2$

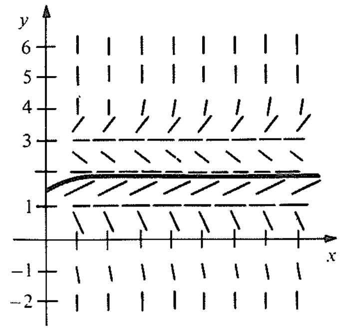
29. $y(1) \approx 2.2469$

31. $y(1) \approx 0.4683$

33. $\lim_{x\to\infty} y(x) = 3$



35. $\lim_{x\to\infty} y(x) = 1$



37. 61

39. $\int h(y)\,dy = -\int (1/g(x))\,dx$

8.6 Linear First-Order Equations

1. $y = 2 + (-3\ln|1 - x| + C)(1 - x)$

3. $y = 1 + C\exp(x^4/4)$

5. $y = -2 + 2\exp(\sin x)$

7. $y = (e^x - e)/x$

9. The equation is $L\dfrac{dI}{dt} + RI = E_0\cos\omega t + E_1$ and has solution

$$I = \frac{E_0}{L}\,\frac{1}{(R/L)^2 + \omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + Ce^{-tR/L} + \frac{E_1}{R}$$

11. $I = E_0C - E_0C\exp(-t/RC)$;
$I \to E_0C$ as $t \to +\infty$.

13. Set $y = .9 \times 2.51 \times 10^6$ and verify the value of t.

15. $6.28 \times 10^5 - (8.28 \times 10^5)\exp(-2.67 \times 10^{-7}t) - (2.01 \times 10^5)\exp(-1.07 \times 10^{-6}t)$

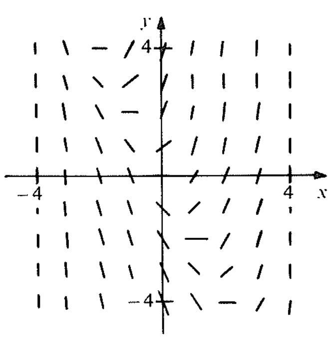
17. 15 seconds; 951 meters.

19. Use separation of variables to get

$$v = \sqrt{mg/\gamma}\,\tanh(\sqrt{\gamma g/m}\,t)$$

21. $\dfrac{FM_0}{M_1^2} - \dfrac{g}{2M_1^2}(M_0^2 + M_1^2)$

23. $y = -2(x + 1) + Ce^x$



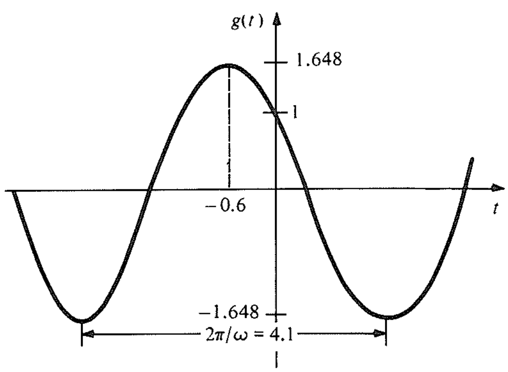
25. If y_1 and y_2 are solutions, prove, using methods of Section 8.2, uniqueness for $y' = P(x)y$ and apply it to $y = y_1 - y_2$. (This is one of several possible procedures.)

27. (a) $w' = (1-n)[Q + Pw]$
(b) $y = \pm 1/(x\sqrt{c - x^2})$

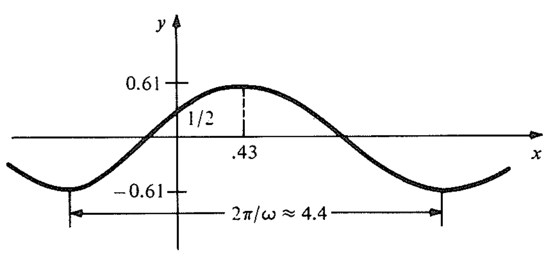
29. (a) $v = \dfrac{F}{\gamma - r} - \dfrac{g(M_0 - rt)}{\gamma - 2r} + C(M_0 - rt)^{\gamma/r - 1}$
where $C = M_0^{1-\gamma/r}\left(\dfrac{gM_0}{\gamma - 2r} - \dfrac{F}{\gamma - r}\right)$ and
where the air resistance force is γv.
(b) At burnout, $v = \dfrac{F}{\gamma - r} - \dfrac{gM_1}{\gamma - 2r} + CM_1^{\gamma/r - 1}$.

Review Exercises for Chapter 8

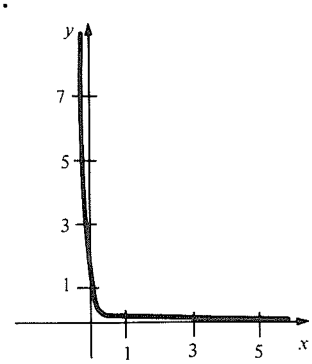
1. $y = e^{3t}$
3. $y = (1/\sqrt{3})\sin\sqrt{3}\,t$
5. $y = (4e^{3t} - 1)/3$
7. $y = 4/(4 - t^4)$
9. $f(x) = e^{4x}$
11. $f(t) = \cosh 2t + \sinh 2t/2$
13. $x(t) = \cos t - \sin t$
15. $x(t) = (\sinh 3t)/3$
17. $y = -\ln(1/e + 1 - e^x)$
19. $x(t) = e^{-4t}$
21. $y = -t$
23. $g(t) = \cos(\sqrt{7/3}\,t - (2/\sqrt{7/3})\sin(\sqrt{7/3}\,t)$; amplitude is $\sqrt{19/7}$; phase is $-\sqrt{3/7}\tan^{-1}(2\sqrt{3/7})$



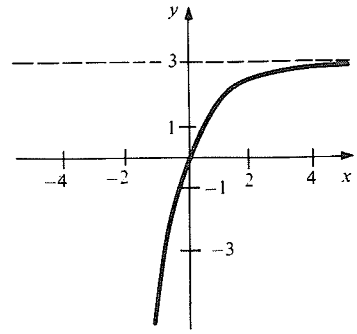
25.



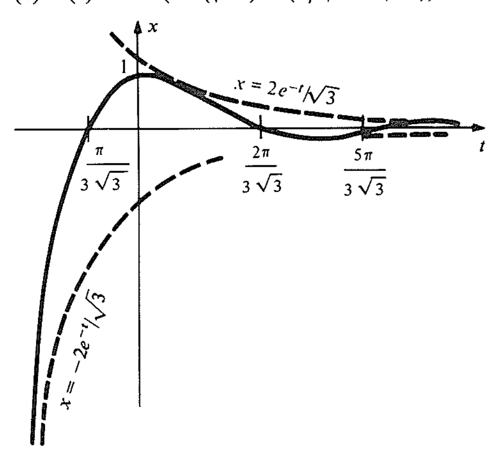
27.



29. $\lim_{t\to\infty} x(t) = 3$

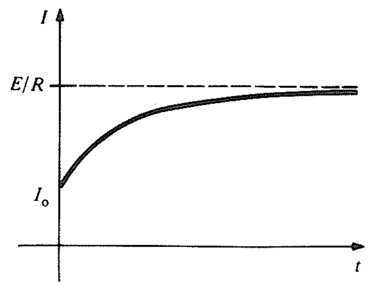


31. $x = e^t$
33. $y = x^2/2 - x - 2e^{-x} + 2$
35. $y(x) = \sinh 5x/\sinh 5$
37. $6x\cosh(3x^2)$
39. $2x/\sqrt{(x^2+1)^2 - 1}$
41. $\cosh 3x/\sqrt{x^2+1} + 3\sinh 3x \sinh^{-1}x$
43. $(-3/\sqrt{9x^2 - 1})\exp(1 - \cosh^{-1}(3x))$.
45. $\tan^{-1}(\sinh x) + C$
47. $(1/3)\tanh^{-1}(x/3) + C$ if $|x| < 3$
$(1/3)\coth^{-1}(x/3) + C$ if $|x| > 3$
49. $x\cosh x - \sinh x + C$
51. $x(t) = \cos\sqrt{2.1/5}\,t$
53. (a) $k = 640$
(b) -6400 newtons
55. (a) $y'' + (\omega^2 - \beta)y = 0$
(c) $x(t) = e^{-t}(\cos(\sqrt{3}\,t) + (1/\sqrt{3}\sin\sqrt{3}\,t))$

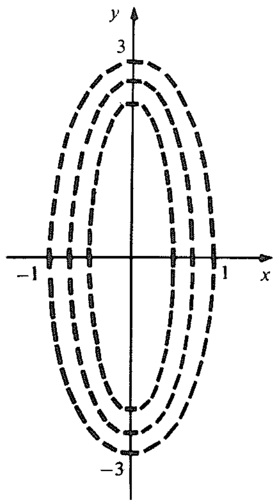


57. 66.4 years
59. 54,150 years
61. 27 minutes
63. (a) 73 years
(b) $S(t) = ke^{-\alpha t}$ where $k = S(0)$

65.



67. (a) $y^2/9 + x^2 = k$, $k = 2C/9$

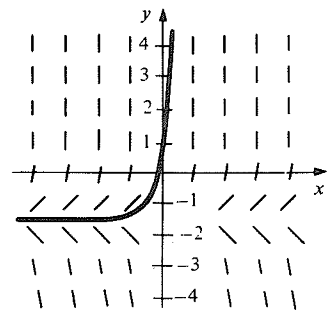


(b) $kx^{1/9}$, $k = e^C$

69. 15.2 minutes, no. [The "no" could be "yes" if you allow a faster addition of fresh water after draining.]

71. $I = 2(3\sin\pi t - \pi\cos\pi t)/(9+\pi^2) + [1 + 2\pi/(9+\pi^2)]e^{-3t}$

73. $y = -4/3 + Ce^{3x}$



75. 1

77. $y = e^x$ is the exact solution; $y(1) = e \approx 2.71828$.

79. $y = -1/(x-1)$ is the exact solution, it is not defined at $x = 1$.

81. $y = Ce^{at} - (b/a)$; the answers are all the same.

83. (a) Verify using the chain rule
(b) Integrate the relation in (a)
(c) Solve for $T = t$; the period is twice the time to go from $\theta = 0$ to $\theta = \theta_0$.

85. (a) $y = \cosh(x + a)$ or $y = 1$.
(b) Area under curve equals arc length.

Chapter 9 Answers

9.1 Volumes by the Slice Method

1. 3π

3. $Ah/3$

5. $2125/54$

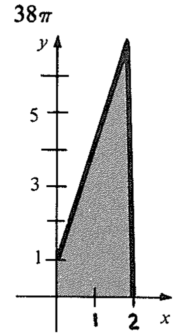
7. $4\sqrt{3}/3$

9. $x_1 = (1 - \sqrt[3]{1/4})h$, $x_2 = (1 - \sqrt[3]{1/2})h$, $x_3 = (1 - \sqrt[3]{3/4})h$

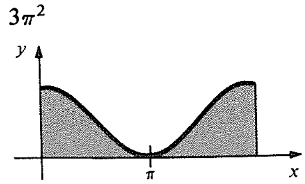
11. $0.022\ \text{m}^3$

13. $1487.5\ \text{cm}^3$

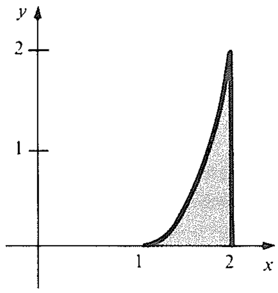
15. 38π



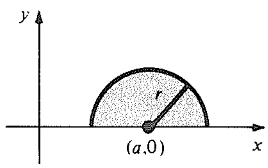
17. $3\pi^2$



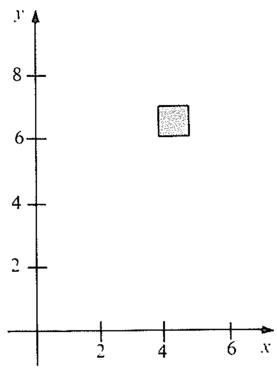
19. $71\pi/105$



21. $\frac{4}{3}\pi r^3$



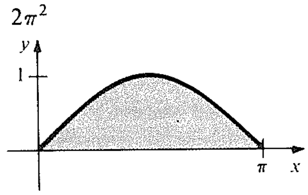
23. 13π



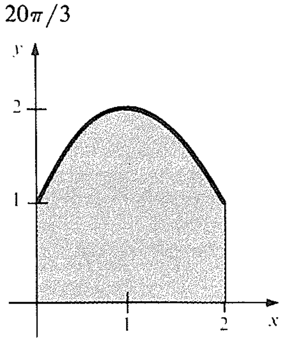
25. 13π (See Exercise 11, Section 9.2 for the figure.)
27. $5\ \text{cm}^3$
29. $V = \pi^2(R + r)(R - r)^2/4$
31. For the two solids, $A_1(x) = A_2(x)$. Now use the slice method.

9.2 Volumes by Shell Method

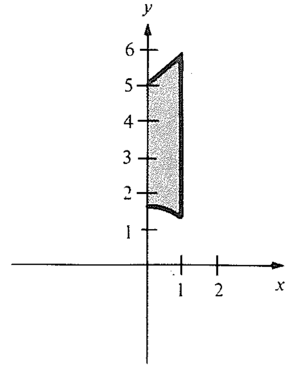
1. $2\pi^2$



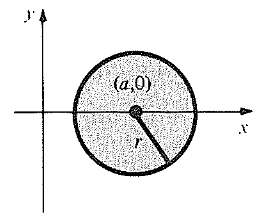
3. $20\pi/3$



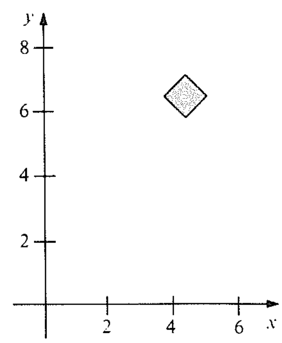
5. $\pi(17 + 4\sqrt{2} - 6\sqrt{3})/3$



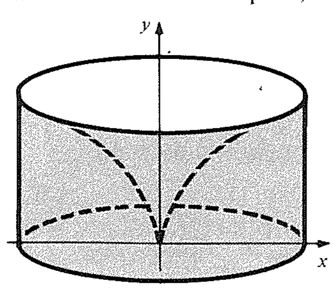
7. $2\pi^2 r^2 a$



9. 9π (See the Figure for Exercise 23, in the left-hand column.)
11. 9π



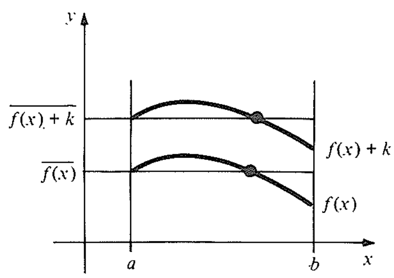
13. $4\pi/5$ (You get a cylinder when this volume is added to that of Example 5, Section 9.1.)



15. $\sqrt{3}\,\pi/2$
17. $24\pi^2$
19. (a) $V = 4\pi r^2 h + \pi h^3/3$
(b) $4\pi r^2$, it is the surface area of a sphere.
21. (a) $2\pi^2 a^2 b$
(b) $2\pi^2 b(2ah + h^2)$
(c) $4\pi^2 ab$
23. $\pi^3/4 - \pi^2 + 2\pi$

9.3 Average Values and the Mean Value Theorem for Integrals

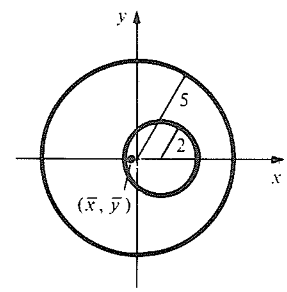
1. $1/4$
3. $\ln\sqrt{5/2}$
5. 2
7. $\pi/4$
9. $\pi/2 - 1$
11. $-2/3\pi$
13. $9+\sqrt{3}$
15. $1/2$
17. 55° F
19. (a) $x^2/3 + 3x/2 + 2$
    (b) The function approaches 2, which is the value of $f(x)$ at $x = 0$.
21. Use the fundamental theorem of calculus and the definition of average value.
23. The average of $[f(x)+k]$ is $k +$ [the averge of $f(x)$].



25. $f(b) - f(a) = \int_a^b f'(x)\,dx = f'(c)\cdot(b-a)$, for some c such that $a < c < b$.
27. $\exp\left[\int_a^b \ln f(x)\,dx/(b-a)\right]$
29. Write $F(x) - F(x_0) = \int_{x_0}^{x} f(s)\,ds$. If $|f(s)| \leqslant M$ on $[a,b]$ (extreme value theorem), $|F(x) - F(x_0)| \leqslant M|x - x_0|$, so given $\epsilon > 0$, let $\delta = \epsilon/M$.

9.4 Center of Mass

1. $$\bar{x} = \frac{m_1x_1 + (m_2+m_3)\left(\dfrac{m_2x_2+m_3x_3}{m_2+m_3}\right)}{m_1 + (m_2+m_3)} = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}.$$
3. Let $M_1 = m_1 + m_2 + m_3$ and $M_2 = m_4$.
5. $\bar{x} = 3$
7. $\bar{x} = 67$
9. $\bar{x} = 1, \bar{y} = 4/3$
11. $\bar{x} = 29/23, \bar{y} = 21/23$
13. (a) $\bar{x} = 1/2, \bar{y} = \sqrt{3}/6$
    (b) $\bar{x} = 3/8, \bar{y} = \sqrt{3}/8$
15. $$\frac{m_1x_1 + (m_2+m_3+m_4)\left[\dfrac{m_2x_2+m_3x_3+m_4x_4}{m_2+m_3+m_4}\right]}{m_1+(m_2+m_3+m_4)} = \frac{m_1x_1+m_2x_2+m_3x_3+m_4x_4}{m_1+m_2+m_3+m_4}$$
17. $\bar{x} = 3(\ln 3)/2, \bar{y} = 26/27$
19. $\bar{x} = 4/(3\pi), \bar{y} = 4/(3\pi)$
21. $\bar{x} = 4/3, \bar{y} = 2/3$
23. Since $x_i \leqslant b$, $\bar{x} = \dfrac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3} \leqslant \dfrac{m_1b+m_2b+m_3b}{m_1+m_2+m_3} = b.$
    Similarly $a \leqslant \bar{x}$. The center of mass does not lie outside the group of masses.
25. Differentiate $\bar{x}$ to get the velocity of the center of mass and use the definitions of P and M.
27. $\bar{x} = -4/21,\ \bar{y} = 0$



29. $\bar{x} = (\sqrt{2}\pi/4 - 1)/(\sqrt{2}-1), \bar{y} = 1/[4(\sqrt{2}-1)]$
31. $\bar{x} = (x_1+x_2+x_3)/3, \bar{y} = (y_1+y_2+y_3)/3$

Supplement to 9.5: Integrating Sunshine

1. The arctic circle receives 1.25 times as much energy as the equator.
3. (a) $F = \sum_{T=0}^{364}{}' \left\{\sqrt{\cos^2 l - \sin^2 D} + \sin l \sin D \cos^{-1}(-\tan l \tan D)\right\}$
   (b) Expressing $\sin D$ in terms of T, the sum in (a) yields
   $$\int_0^{365}\left\{\sqrt{\cos^2 l - \sin^2\alpha\cos^2(2\pi T/365)} + \sin l \sin\alpha\cos(2\pi T/365) \times \cos^{-1}\left[\frac{-\tan l \sin\alpha\cos(2\pi T/365)}{\sqrt{1-\sin^2\alpha\cos^2(2\pi T/365)}}\right]\right\}dT.$$
   This is an "elliptic integral" which you cannot evaluate.
5. $\pi\sin l \sin D$
7. 0.294; it is consistent with the graph ($T = 16.5$; about July 7).

9.5 Energy, Power, and Work

1. 1,890,000 joules
3. $360 + 96/\pi$ watt-hours
5. $3/2$
7. 0.232
9. 98 watts
11. (a) $18t^2$ joules
(b) 360 watts
13. 1.5 joules
15. (a) 45,000 joules
(b) 69.3 meters/second
17. 41,895,000 joules
19. 125,685,000 joules
21. 0.15 joules
23. 1.48×10^8 joules

Review Exercises for Chapter 9

1. (a) $\pi^2/2$
(b) $2\pi^2$
3. (a) $3\pi/2$
(b) $2\pi(2\ln 2 - 1)$
5. $64\sqrt{2}\,\pi/81$
7. $72\pi/5$
9. $5/4$
11. 1
13. 6
15. Apply the mean value theorem for integrals.
17. $1/3, 4/45, 2\sqrt{5}/15$
19. $1, (e^2-5)/4, \sqrt{e^2-5}/2$
21. $3/2, 1/4, 1/2$
23. (a) $\pi\int_a^b \rho(x)[f(x)]^2\,dx$
(b) $(14\pi/45)$ grams
25. $\bar{x} = 5/3, \bar{y} = 40/9$
27. $\bar{x} = 1/4(2\ln 2 - 1), \bar{y} = 2(\ln 2 - 1)^2/(2\ln 2 - 1)$
29. $\bar{x} = 27/35, \bar{y} = -12/245$
31. (a) $7500 - 2100e^{-6}$ joules
(b) $\frac{1}{6}(125 - 35e^{-6})$ watts
33. $120/\pi$ joules
35. $\rho g\pi\int_0^a x^2[f(a) - f(x)]f'(x)\,dx$; the region is that under the graph $y = f(x)$, $0 \leqslant x \leqslant a$, revolved about the y-axis.
37. (a) The force on a slab of height $f(x)$ and width dx is $dx\int_0^{f(x)}\rho g y\,dy = \frac{1}{2}\rho g[f(x)]^2\,dx$. Now integrate.
(b) If the graph of f is revolved about the x axis, the total force is $\rho g/2\pi$ times the volume of the solid.
(c) $\frac{2}{3}\rho g \times 10^6 = 6.53 \times 10^9$ Newtons.
39. (a) $\left\{\frac{1}{b-a}\sum_{j=1}^{n}\left[k_j - \frac{1}{b-a}\sum_{i=1}^{n}k_i(t_i - t_{i-1})\right]^2(t_j - t_{j-1})\right\}^{1/2}$
(b) $\left\{\frac{1}{n}\sum_{j=1}^{n}\left[k_j - \frac{1}{n}\sum_{i=1}^{n}k_i\right]^2\right\}^{1/2}$
(c) Show that if the standard deviation is 0, $k_i - \mu = 0$, which implies $k_i = \mu$.
(d) $\left\{\frac{1}{n}\sum_{j=1}^{n}\left[a_i - \frac{1}{n}\sum_{i=1}^{n}a_i\right]^2\right\}^{1/2}$
(e) All numbers in the list are equal.
41. Let $g(x) = f(\alpha x) - c$. Adjust α so g has zero integral. Apply the mean value theorem for integrals to g. (There may be other solutions as well.)
43. The average value of the logarithmic derivative is $\ln[f(b)/f(a)]/(b-a)$.

Chapter 10 Answers

10.1 Trigonometric Integrals

1. $(\cos^6 x)/6 - (\cos^4 x)/4 + C$
3. $3\pi/4$
5. $(\sin 2x)/4 - x/2 + C$
7. $1/4 - \pi/16$
9. $(\sin 2x)/4 - (\sin 6x)/12 + C$
11. 0
13. $-1/(3\cos^3 x) + 1/(5\cos^5 x) + C$
15. The answers are both $\tan^{-1}x + C$
17. $\sqrt{x^2-4} - 2\cos^{-1}(2/x) + C$
19. $(1/2)(\sin^{-1}u + u\sqrt{1-u^2}) + C$
21. $\sqrt{4+s^2} + C$
23. $(-1/3)\sqrt{4-x^2}(x^2+8) + C$
25. $(1/2)\sinh^{-1}((8x+1)/\sqrt{15}) + C$
27. $\sqrt{\left(x+\frac{1}{6}\right)^2 - \frac{13}{36}}$
$$-\frac{1}{6\sqrt{3}}\ln\left|\frac{6x+1}{\sqrt{13}} + \sqrt{\frac{(6x+1)^2}{13} - 1}\right| + C$$
29. $1, 0, 1/2, 0, 3/8, 0, 5/16$.
31. $\bar{x} = (\sqrt{5} - \sqrt{2})/\ln((\sqrt{5}+2)/(\sqrt{2}+1)) - 1$
$\bar{y} = (\tan^{-1}2 - \pi/4)/[2\ln((\sqrt{5}+2)/(\sqrt{2}+1))]$
33. 125
35. $\sqrt{3}, 9\sqrt{2}/4$
37. (a) Differentiate $[S(t)]^3$ and integrate the new expression.
(b) $[3(-t\cos t + \sin t + t/8 - (1/32)\sin 4t)]^{1/3}$
(c) Zeros at $t = n\pi$, n a positive integer. Maxima occur when n is odd.

10.2 Partial Fractions

1. $(1/125)\{4\ln[(x^2+1)/(x^2-4x+4)] + (37/2)\tan^{-1}x + (15x-20)/2(1+x^2) - 5/(x-2)\} + C$
3. $5/4 - 3\pi/8$
5. $(1/5)\{\ln(x-2)^2 + (3/2)\ln(x^2+2x+2) - \tan^{-1}(x+1)\} + C$

7. $2 + (1/3)\ln 3 + (2/\sqrt{3})(\tan^{-1}(5/\sqrt{3}) - \tan^{-1}(3\sqrt{3}))$
9. $(1/8)\ln((x^2 - 1)/(x^2 + 3)) + C$
11. $(1/2)\ln(5/2)$
13. $2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$
15. $\frac{3}{8}(x^2 + 1)^{4/3} + C$
17. $-2/(1 + \tan(x/2)) + C$
19. $\pi/16 - (1/4)\ln|(1 + \tan(\pi/8))(1 + 2\tan(\pi/8) - \tan^2(\pi/8))| \approx -0.017$
21. $\pi\ln(225/176)$
23. $3(1 + x)^{2/3}/4 + (3/4\sqrt[3]{4})\ln|\sqrt[3]{4}(1 + x)^{2/3} + (2 + 2x)^{1/3} + 1| - (1/2\sqrt[6]{432})\tan^{-1}[(2(4 + 4x)^{1/3} + \sqrt[3]{2}/\sqrt[6]{108}] + C$
25. (a) $\frac{1}{20}\ln\left|\frac{x - 80}{x - 60}\right| = kt + \frac{1}{20}\ln\frac{4}{3}$
(b) $x = \dfrac{80(1 - e^{-20kt})}{\frac{4}{3} - e^{-20kt}}$
(c) 26.2 kg
27. (a) Using the substitution, we get

$$(q/m)\int u^{p+q-1}x^{r-m+1}\,du.$$

(b) If $r - m + 1 = mk$, the integral in (a) becomes

$$(q/m)\int u^{p+q-1}(u^q - b)^k\,du$$

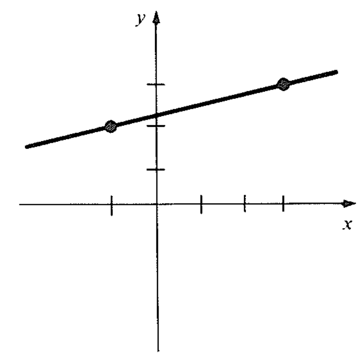
which is an integral of a rational function of u.

10.3 Arc Length and Surface Area

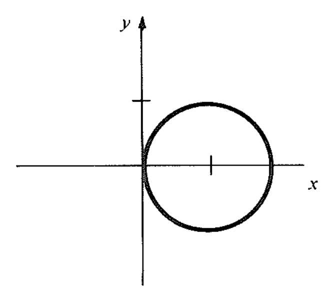
1. $92/9$
3. $14/3$
5. $\int_a^b \sqrt{1 + n^2x^{2n-2}}\,dx$
7. $\int_0^1 \sqrt{1 + \cos^2 x - 2x\sin x\cos x + x^2\sin^2 x}\,dx$
9. $\sqrt{5} + \sqrt{2} + \sqrt{10}$
11. $\sqrt{5} + \sqrt{2} + \sqrt{17}$
13. $(\pi/6)(13^{3/2} - 5^{3/2})$
15. $2654\pi/9$
17. $2\pi(\sqrt{2} + \ln(1 + \sqrt{2}))$
19. $\pi[(3^{4/3} + 1/9)^{3/2} - (10/9)^{3/2}]$
21. $2\sqrt{2}\,\pi$
23. $\pi(6\sqrt{2} + 4\sqrt{5})$
25. $(1/27a^2)[(4 + 9a^2(1 + b))^{3/2} - (4 + 9a^2b)^{3/2}]$; the answer is independent of c.
27. $\int_{-1}^2 \sqrt{1 + 36x^4}\,dx \approx 19$
29. (a) $\int_0^{\pi/2} \sqrt{5 + \sec^4 x + 4\sec^2 x}\,dx$
(b) $2\pi\int_0^{\pi/2} (\tan x + 2x)\sqrt{5 + \sec^4 x + 4\sec^2 x}\,dx$
31. (a) $\int_1^2 \sqrt{1 + (1 - 1/x^2)^2}\,dx$
(b) $2\pi\int_1^2 (1/x + x)\sqrt{1 + (1 - 1/x^2)^2}\,dx$
33. Dividing the curve into 1 mm segments and revolving these, we get about 16 cm^2.
35. Use $|\sin\sqrt{3}\,x| \leqslant 1$ to get $L \leqslant \int_0^{2\pi}\sqrt{1 + 3}\,dx = 4\pi$.
37. Estimate each integral numerically.
39. $2\pi\int_a^b [1/(1 + x^2)]\left(\sqrt{1 + 4x^2/(1 + x^2)^4}\right)dx$; the integrand is $\leqslant \sqrt{5}/(1 + x^2)$.
41. (a) $\pi(a + b)\sqrt{1 + m^2}(b - a)$
(b) Use part (a).

10.4 Parametric Curves

1. $y = (1/4)(x + 9)$

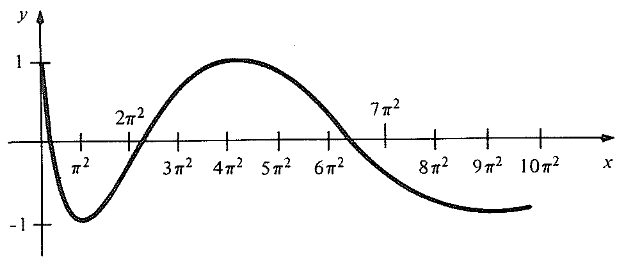


3. $1 = (x - 1)^2 + y^2$

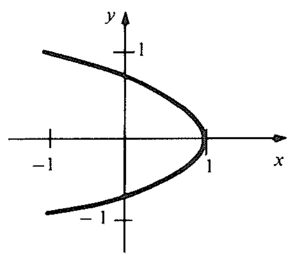


5. $x = t, y = \pm\sqrt{1 - 2t^2}$ or $x = \cos t/\sqrt{2}, y = \sin t$
7. $x = t, y = 1/4t$.
9. $x = t, y = t^3 + 1$.
11. $x = t, y = \cos(2t)$.
13. $y = (1/3)(x + 3/2)$
15. $y = 1/2$
17. $(13, -7)$

19. $y = \cos\sqrt{x}$ $(x \geqslant 0)$, horizontal tangents at $t = n\pi$, n a nonzero integer. The slope is $-1/2$ at $t = 0$ although the curve ends.



21. $y^2 = (1 - x)/2$, vertical tangents at $t = n\pi$, n an integer



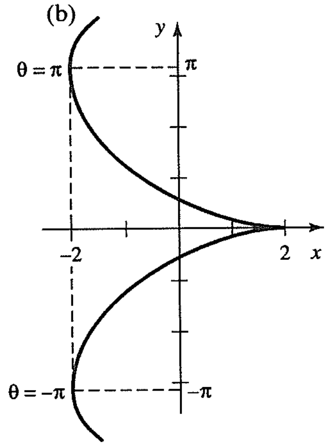
23. $(13^{3/2} - 8)/27$

25. $(1/2)[\sqrt{5} + (1/2)\ln(2 + \sqrt{5})]$

27. (a) Calculate the speed directly to show it equals $|a|$.
(b) Calculate directly to get $|a|(t_1 - t_0)$

29. (a) $y = -x/2 + \pi/2 - 1$

(b)



(c) $\int_0^{\pi} \sqrt{5 - 3\cos^2\theta - 2\cos\theta}\, d\theta$

31. 5

33. (a) $\dot{x} = k(\cos\omega t - \omega t \sin\omega t)$;
$\dot{y} = k(\sin\omega t + \omega t \cos\omega t)$.
(b) $k\sqrt{1 + \omega^2 t^2}$
(c) $2mk\omega$

35. (a) $x = t + (1 + 4t^2)^{-1/2}$,
$y = t^2 + 2t(1 + 4t^2)^{-1/2}$
(b) $x = \pm(1/2)\sqrt{1/(x^2 - y) - 1} + \sqrt{x^2 - y}$.

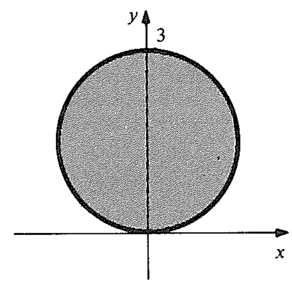
37. (a) We estimated about 338 miles.
(b) We estimated about 688 miles.
(c) It would probably be longer.
(d) The measurement would depend on the definition and scale of the map used.
(e) From the *World Almanac and Book of Facts* (1974), Newspaper Enterprise Assoc., New York, 1973, p. 744, we have coastline: 228 miles, shoreline: 3,478 miles.

10.5 Length and Area in Polar Coordinates

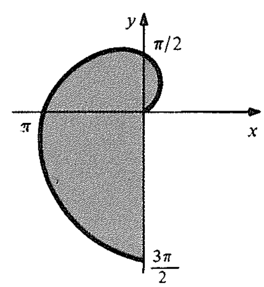
1. 24

3. $(4/3)(13^{3/2} - 8)$

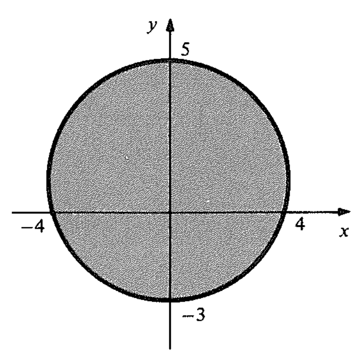
5. $9\pi/4$



7. $9\pi^3/16$

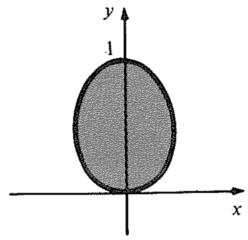


9. $33\pi/2$

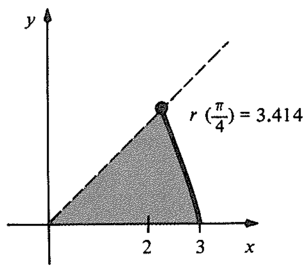


11. $2\pi r$

13. $s = \int_{-\pi/2}^{\pi/2} \sqrt{\sec^2(\theta/2)/4 + \tan^2(\theta/2)}\, d\theta$
$A = 2 - \pi/2$



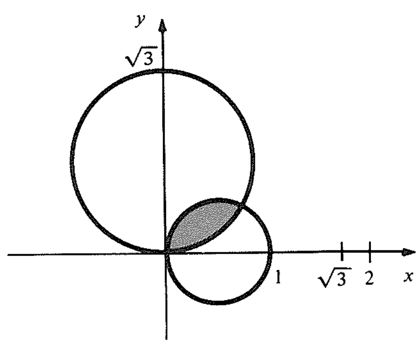
15. $s = \int_0^{\pi/4} \sqrt{\sec^2\theta \tan^2\theta + \sec^2\theta + 4\sec\theta + 4}\, d\theta$
$A = 1/2 + \pi/2 + \ln(3 + 2\sqrt{2})$



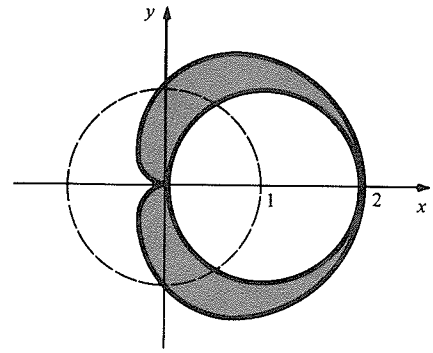
17. $s =$
$\int_0^{\pi/2} \sqrt{(1 + \cos\theta - \theta\sin\theta)^2 + \theta^2(1 + 2\cos\theta + \cos^2\theta)}\, d\theta$
$A = (1/2)[\pi^3/16 + \pi^2/2 - 4 - \pi/8]$

19. $s = \int_0^{\pi/2} \sqrt{(5 + 4\sin 2\theta)/(1 + 2\sin 2\theta)}\, d\theta$
$A = \pi/2$

21. $A = (1/4)(5\pi/6 - \sqrt{3})$
$L = (2 + \sqrt{3})\pi/6$



23. $A = \pi/2$
$L = 2\pi + 8$



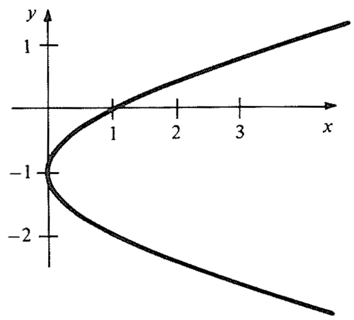
25. $\sqrt{2}\,(e^{2(n+1)\pi} - e^{2n\pi})$

27. (a) Use $x = a\cos t$, $y = b\sin t$, where $T = 2\pi$.
(b) Substitute into the given formula.

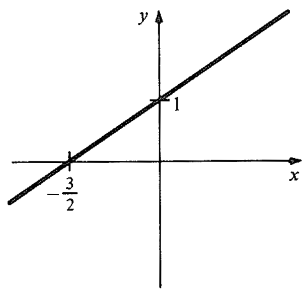
Review Exercises for Chapter 10

1. $\sin^3 x + C$
3. $(\cos 2x)/4 - (\cos 8x)/16 + C$
5. $(1 - x^2)^{3/2} - \sqrt{1 - x^2} + C$
7. $4(x/4 - \tan^{-1}(x/4)) + C$
9. $(2\sqrt{7}/7)\tan^{-1}[(2x + 1)/\sqrt{7}] + C$
11. $\ln|(x + 1)/x| - 1/x + C$
13. $(1/2)[\ln|x^2 + 1| + 1/(x^2 + 1)] + C$
15. $\tan^{-1}(x + 2) + C$
17. $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$
19. $-(1/2a)\cot(ax/2) - (1/6a)\cot^3(ax/2) + C$
21. $\ln|\sec x + \tan x| - \sin x + C$
23. $(\tan^{-1}x)^2/2 + C$
25. $(1/3\sqrt[3]{9})[\ln|x - \sqrt[3]{9}| - \ln\sqrt{x^2 + \sqrt[3]{9}\,x + 3\sqrt[3]{3}}$
$+ \sqrt{3}\tan^{-1}((2x/\sqrt[3]{9} + 1)/\sqrt{3})] + C.$
27. $2\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}} + C$
29. $x - \ln(e^x + 1) + C$
31. $(-1/4)[(2x^2 - 1)/(x^2 - 1)^2] + C$
33. $-(1/10)\cos 5x - (1/2)\cos x + C$
35. $\ln\sqrt{x^2 + 1} + C$
37. $2e^{\sqrt{x}} + C$
39. $\frac{1}{2}\ln 2$
41. $\frac{1}{2}\ln(x^2 + 3) + C$
43. $x^4\ln x/4 - x^4/16 + C$
45. $\frac{1}{4}[(\ln 6 + 5)^4 - (\ln 3 + 5)^4] \approx 186.12$
47. $(1/4)\sinh 2 - 1/2$
49. 0
51. $(733^{3/2} - 4^{3/2})/243$
53. $59/24$
55. $\pi(5^{3/2} - 1)/6$
57. ≈ 31103

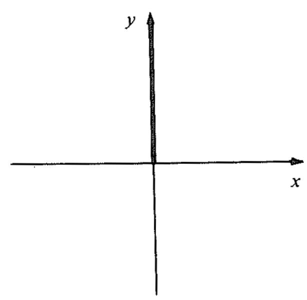
59. $x = (y+1)^2$



61. $y = 2x/3 + 1$



63. $x = 0, y \geqslant 0$



65. $y = 3x/4 + 5/4$

67. $(1/8)(\sqrt{257} \cdot 16 + \ln|\sqrt{257} + 16|)$

69. $L = (1/3)[(\pi^2/4 + 4)^{3/2} - 8]$
$A = \pi^5/320$

71. $L = \int_0^{\pi} \sqrt{(5/4) + \cos 2\theta + 3\sin^2 2\theta}\, d\theta$
$A = 3\pi/8$

73. $L = 5\sqrt{2}$
$A = 315\pi/256 + 9/4$

75. $b_2 = 1$, all others are zero.

77. $a_3 = 1$, all others are zero.

79. $a_4 = 3$, all others are zero.

81. $a_0 = 1$, $a_2 = -1/2$, all others are zero.

83. (a) $(1/k_2)\ln[N_0(k_1N(t) - k_2)/N(t)(k_1N_0 - k_2)]$
(b) $N(t) = k, N_0/[k_1N_0(1 - e^{k_2t}) + k_2e^{k_2t}]$
(c) The limit exists if $k_2 > 0$ and it equals k_2/k_1.

85. Use $(\cos\phi)\,d\phi = (\sin\phi_m)(\cos\beta)\,d\beta$ and substitute.

87. $a^{-1/2}\ln|2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c}\,| + C$,
$a > 0$
$(-a)^{-1/2}\sin^{-1}[(-2ax - b)/\sqrt{b^2 - 4ac}\,] + C$,
$a < 0$

89. (a) $b - a + (b^{n+1} - a^{n+1})/(n+1)$ if $n \neq -1$. If $n = -1$, we have $b - a + \ln(b/a)$.
(b) $n = 0$: $L = b - a$; $n = 1$: $L = \sqrt{2}\,(b - a)$;
$n = 2$: see Example 3 of Section 10.3;
for $n = (2k+3)/(2k+2)$, $k = 0, 1, 2, 3, \ldots$

$$L = \left\{ \frac{n^{1/(1-n)}}{n-1}(1 + n^2x^{2n-2})^{3/2} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{2j+3}(1 + n^2x^{2n-2})^j \right\} \Bigg|_{x=a}^{x=b};$$

$n = \frac{3}{2}: L = \frac{1}{27}[(4+9b)^{3/2} - (4+9a)^{3/2}]$.

(c) Around the x-axis, we have

$$\pi\left[b - a + \frac{2(b^{n+1} - a^{n+1})}{n+1} + \frac{b^{2n+1} - a^{2n+1}}{2n+1} \right]$$

if $n \neq -1$ or $-1/2$. For $n = -1$ we have

$$\pi\left[b - a + 2\ln(b/a) - (a^{-1} - b^{-1}) \right].$$

For $n = -1/2$ we have

$$\pi\left[b - a + 4\sqrt{b} - 4\sqrt{a} + \ln(b/a) \right].$$

Around the y-axis we have

$$\pi\left[b^2 - a^2 + \frac{2(b^{n+2} - a^{n+2})}{n+2} \right]$$

if $n \neq -2$. For $n = -2$, we have $\pi[b^2 - a^2 + 2\ln(b/a)]$.
(d) $A_x = 2\pi L$ (from 89(b)) $+ A_x$ (from 88(d))
$A_y = A_y$ (from 88(d))
Some answers from 88(d) needed here are:
88(d).

$n = 0$; $A_x = 2\pi(b - a)$

$n = 1$; $A_x = \sqrt{2}\,\pi(b^2 - a^2)$

$$n = 2;\ A_x = \frac{\pi}{32}\left[(1 + 8x^2)2x\sqrt{1 + 4x^2} - \ln(2x + \sqrt{1 + 4x^2}\,) \right] \Bigg|_{x=a}^{x=b}$$

$$n = 3;\ A_x = \frac{\pi}{27}(1 + 9x^4)^{3/2} \Big|_{x=a}^{x=b}$$

$n = (2k+3)/(2k+1)$; $k = 0, 1, 2, 3, \ldots$

$$A_x = \frac{2\pi}{n-1} n^{(1+n)/(1-n)}(1 + n^2x^{2n-2})^{3/2} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{2j+3}(1 + n^2x^{2n-2})^j$$

$n = 0$; $A_y = \pi(b^2 - a^2)$

$n = 1$; $A_y = \sqrt{2}\,\pi(b^2 - a^2)$

$$n = 2;\ A_y = \frac{\pi}{6}\left[(1 + 4b^2)^{3/2} - (1 + 4a^2)^{3/2} \right]$$

$n = (k+2)/(k+1)$; $k = 0, 1, 2, 3, \ldots$;

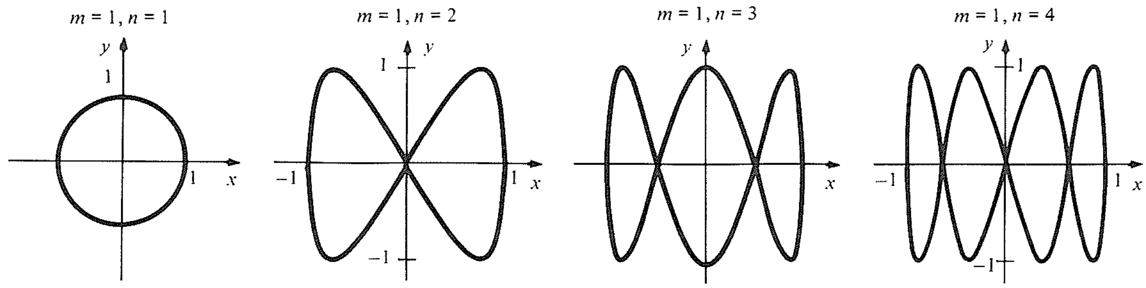
$$A_y = \frac{2\pi}{n-1} n^{2/(1-n)}(1 + n^2x^{2n-2})^{3/2}$$

$$\times \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{2j+3}(1 + n^2x^{2n-2})^j \Big|_a^b$$

91. (a) $2\pi \int_{\alpha}^{\beta} r\sin\theta\sqrt{r^2 + (r')^2}\, d\theta$

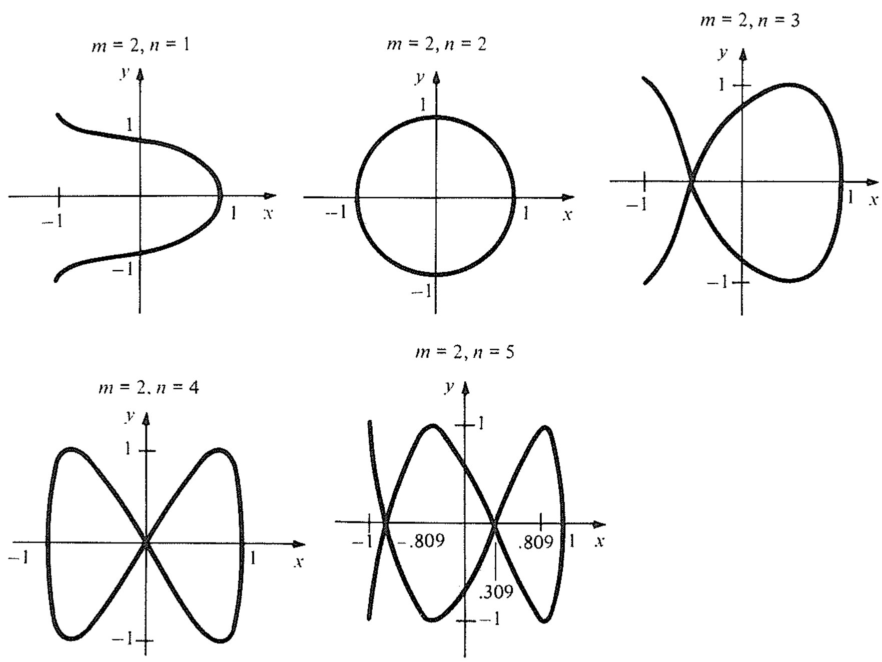
(b) $2\pi \int_{-\pi/4}^{\pi/4} \cos 2\theta \sin\theta \sqrt{1 + 3\sin^2 2\theta}\, d\theta$

93. (a)

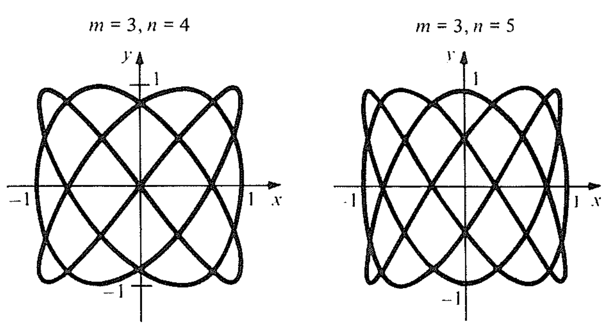


(b) Each curve will consist of n loops for n odd or even.

(c)



(d)



95. The last formula is the average of the first two.

Chapter 11 Answers

11.1 Limits of Functions

1. Choose δ less than 1 and $\varepsilon/(1 + 2|a|)$.

3. Write $x^3 + 2x^2 - 45 =$
$[(x-3)+3]^3 + 2[x-3]^2 - 45$ and expand.

5. e^3 **7.** 5

9. -4 **11.** 6

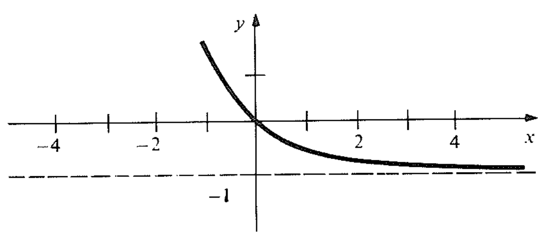
13. $A = 1/\sqrt[3]{\varepsilon}$ **15.** $A = -\ln\varepsilon/3$

17. -2 **19.** $2/3$

21. $3/5$ **23.** $1/2$

25. 0. Consider $\sqrt{x^2+a^2} - x$ as the difference between the hypotenuse and a leg of a right triangle. As x gets large, the difference becomes small.

27. $y = -1$ is a horizontal asymptote.

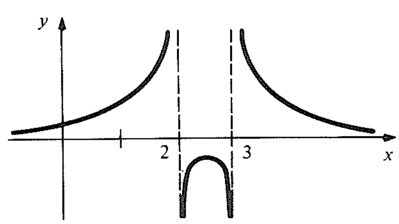


29. $+\infty$ **31.** $+\infty$

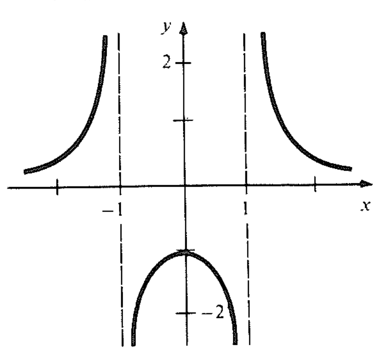
33. $+\infty$ **35.** $-\infty$

37. -1 **39.** -1

41. Vertical asymptotes at $x = 2, 3$. Horizontal asymptote at $y = 0$.



43. Vertical asymptotes at $x = \pm 1$, horizontal asymptote at $y = 0$.



45. (a) Given ε, the A for g is the same as for f (as long as $|g(x)| \leqslant |f(x)|$ for $x \geqslant A$).
(b) 0

47. $7/9$ **49.** $3/2$

51. $4/5$ **53.** $2n + 1$

55. $16/17$

57. $+\infty$

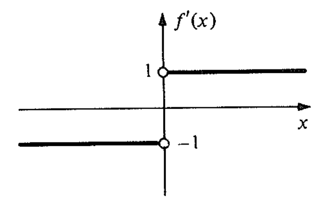
59. $-\infty$

61. $y = 0$ is a horizontal asymptote; $x = -1$, $x = 1$ are vertical asymptotes.

63. $y = \pm 1$ are horizontal asymptotes.

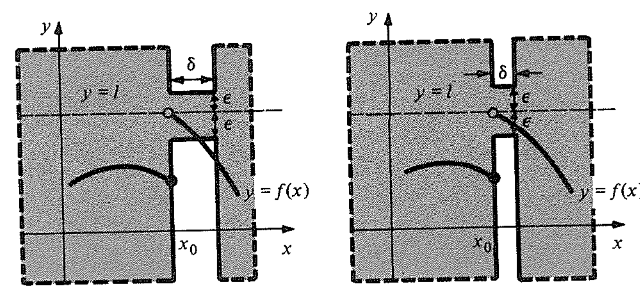
65. If $f(x) = a_n x^n + \cdots$ and $g(x) = b_n x^n + \cdots$, show that $a_n/b_n = l$. If $l = \pm\infty$, then $\lim_{x\to-\infty} f(x)$ can be $\pm\lim_{x\to\infty} f(x)$.

67. (a) $f'(x) = -1$ for $x < 0$, $f'(x) = 1$ for $x > 0$, $f'(0)$ is not defined.

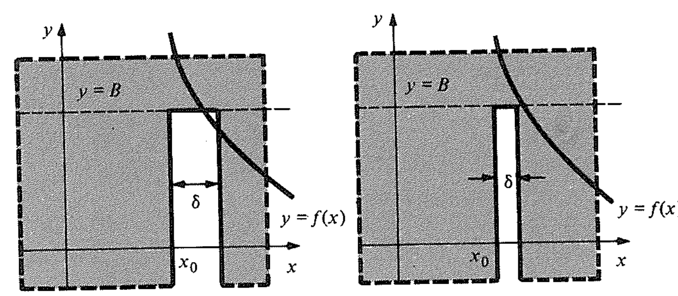


(b) As $x \to 0-$, the limit is -1, while as $x \to 0+$, we get 1.
(c) No.

69. (a)



(*i*) a poor δ (*ii*) a good δ



(*i*) a poor δ (*ii*) a good δ

71. N_0, which means that the population in the distant future will approach an equilibrium value N_0.

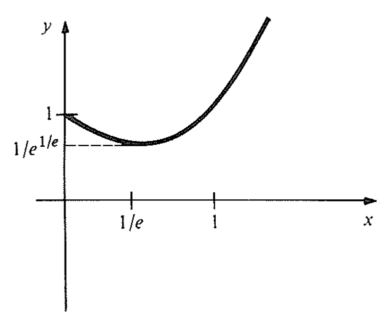
73. Use the laws of limits

75. Write $af(x) + bg(x) - aL - bM =$
$a[f(x) - L] + b[g(x) - M]$

77. Repeat the argument given, using $|x - x_0| < \delta$ in place of $x_0 < x < x_0 + \delta$.
79. Given $B > 0$, let $\varepsilon = 1/B$. Choose δ so that $|1/f(x)| < \varepsilon$ when $|x - x_0| < \delta$; then $|f(x)| > B$ for $|x - x_0| < \delta$.
81. If $x \geqslant A$, $y \leqslant \delta$ where $\delta = 1/A$, $y = 1/x$.

11.2 L'Hôpital's Rule

1. 108
3. 2
5. $-9/10$
7. $-4/3$
9. ∞
11. 0
13. 0
15. 0
17. 1
19. 0
21. 0
23. 0
25. 0
27. does not exist (or is $+\infty$)
29. 0
31. $1/24$
33. 0
35. $1/120$
37. 0
39. The slope of the chord joining $(g(a), f(a))$ to $(g(b), f(b))$ equals the slope of the tangent line at some intermediate point.
41.



43. (a) $1/2$
(b) 1
(c) yes

11.3 Improper Integrals

1. 3
3. $e^{-5}/5$
5. $(\ln 3)/2$
7. $\pi/2$
9. Use $1/x^3$
11. Use e^{-x}
13. Use $1/\sqrt{3x}$ on $[1, \infty)$
15. Use $1/x$
17. $3\sqrt[3]{10}$
19. 2
21. Diverges
23. Converges
25. Converges
27. Converges
29. Converges
31. Diverges
33. Converges
35. Converges
37. Diverges
39. Diverges
41. $k > 1$ or $k = 0$
43. ≈ 2.209
45. $6\sqrt{3}$ hours
47. $\pi e^{-20}/2$
49. $\ln(2/3)$
51. Follow the method of Example 11.
53. (a) Change variables
(b) Use the comparison test. (Compare with $e^{x/2}$ for $x \leqslant -1$ and $e^{-x/2}$ for $x \geqslant 1$.)
55. (a) π
(b) $(p-1)(q-1) < 0$.
57. $f(x) = f(0) + \int_0^x f'(s)\,ds$; the integral converges.

11.4 Limits of Sequences and Newton's Method

1. n must be at least 6.
3. $\lim_{n\to\infty}(a_n) = 2$
5. $0, -1, 4 - 2\sqrt{2}, 9 - 2\sqrt{3}, 12$
7. $1/7, 1/14, 1/21, 1/28, 1/35, 1/42$
9. The sequence is $1/2$ for all n.
11. $N \geqslant 3/\varepsilon$
13. $n \geqslant 3/2\varepsilon$
15. 3
17. -3
19. 4
21. 0
23. 0
25. The limit is 1.
27. The limit is 1.
29. 0
31. 0
33. 0
35. (a) $x = 0.523148$ is a root.
(b) $x = -0.2475, 7.7243$
37. $x = 1.118340$ is a root.
39. One root is $x = 4.493409$.
41.

	$\alpha = 2$	$\alpha = 3$	$\alpha = 5$
λ_1	1.1656	1.3242	1.4320
λ_2	4.6042	4.6407	4.6696
λ_3	7.7899	7.8113	7.8284

43. $1/e \approx 0.36788$
45. $a_n = 2^{2^{n-1}}$
47. Use the definition of limit and let ε be a.
49. $1, 1/2, 1/4, 1/8, 1/16, \ldots, 1/(2^n), \ldots$; the limit is 0.
51. The limit does not exist.
53. $3/4$
55. (a) For any $A \geqslant 0$ there is an N such that $a_n \geqslant A$ if $n \geqslant N$, (b) let $N = 16A$.
57. (a) Assume $\lim_{n\to\infty} b_n < L$ and look at
$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n.$$
(b) Write $b_n - L = (b_n - a_n) + (a_n - L) \leqslant (c_n - a_n) + (a_n - L)$.
59. (a) Below about $a = 3.0$, iterates converge to a single point; at $a \approx 3.1$, they oscillate between two points; as a increases towards 4, the behavior gets more complicated.
(b), (c) See the references on p. 548.

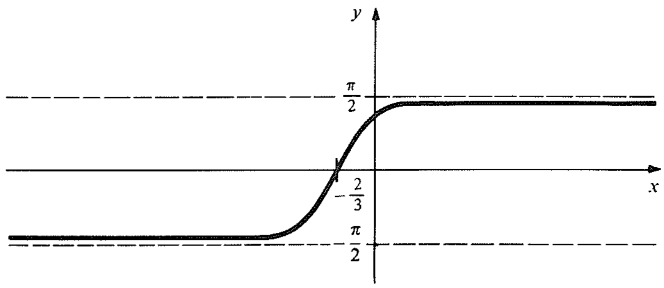
11.5 Numerical Integration

1. 2.68; actual value is $8/3$
3. ≈ 0.13488
5. ≈ 0.3246
7. ≈ 1.464

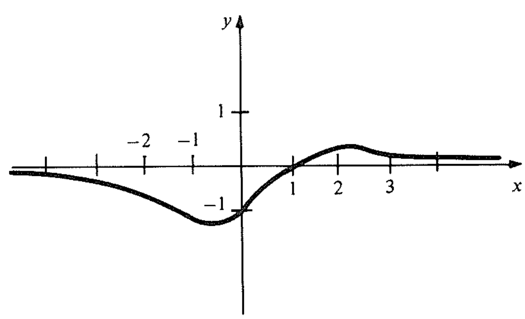
9. ≈ 2.1824
11. Evaluation gives $a(x_2^3 - x_1^3)/3 + b(x_2^2 - x_1^2)/2 + c(x_2 - x_2)$. Since $f''''(x) = 0$, Simpson's rule gives the exact answer. The error for the trapezoidal rule depends on $f''(x)$ and is nonzero.
13. 180, 9
15. 158 seconds
17. The first 2 digits are correct.

Review Exercises for Chapter 11

1. Choose δ to be $\min(1, \varepsilon/4)$.
3. Choose δ to be $\min(1,\varepsilon/5)$; $\min(1,\varepsilon/3)$ is also correct.
5. $\tan(-1)$ 7. 1
9. 0 11. ∞
13. 0 15. 0
17. $y = \pm\pi/2$ are horizontal asymptotes.



19. $y = 0$ is a horizontal asymptote.



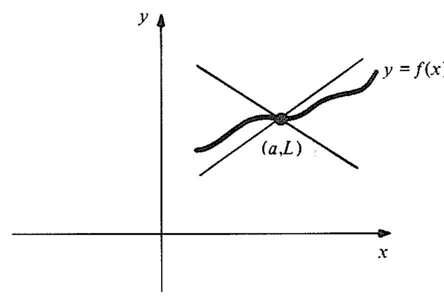
21. $1/4$ 23. 0
25. 0 27. 5
29. $-1/6$ 31. $\sec^2(3)$
33. 1 35. 0
37. 0 39. 1
41. e^2 43. 0
45. Converges to 1 47. Diverges
49. Converges to 2 51. Converges to $5/3$
53. Converges to $-1/4$ 55. $2\pi/3\sqrt{3}$
57. $\pi/4$ 59. 32,768
61. e^8 63. 0
65. 1 67. $\tan 3$
69. Does not exist 71. $-2/5$
73. 1 75. 0
77. -1.35530 (the only real root)
79. 1.14619
81. 2.31992 83. 50.154
85. Both 87. $1/\sqrt{x}$
89. (b)

$$\lim_{h\to 0}\{[f(x_0+2h) - 3f(x_0+h) + 3f(x_0) - f(x_0-h)]/h^3\}$$

91. 1
93. S_n is the Riemann sum for $f(x) = x + x^2$.
95. The exact amount is

$$P(e^r + e^{364r/365} + \cdots + e^{r/365})$$

97. (a)



(c) Choose $\delta = \varepsilon/2m$, (or h, whichever is smallest).

101. (a) Use the definition of N
(b) Use the quotient rule
(c) $|N(x) - \bar{x}| \leqslant (Mq/p^2)|x - \bar{x}|^2$
(d) 5

Chapter 12 Answers

12.1 The Sum of an Infinite Series

1. $1/2, 5/6, 13/12, 77/60$
3. $2/3, 30/27, 38/27, 130/81$
5. $7/6$ 7. 7
9. \$40,000 11. $1/12$
13. $16/27$ 15. $81/2$
17. $3/2$ 19. $64/9$
21. $\sum 1$ diverges and $\sum 1/2^i$ converges
23. 7 25. Diverges
27. Diverges 29. Diverges
31. Reduce to the sum of a convergent and a divergent series.
33. Let $a_i = 1$ and $b_i = -1$.
35. (a) $a_1 + a_2 + \cdots + a_n = (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_{n+1} - b_n) = b_{n+1} - b_1$ (see Section 4.1).
(b) 1

37. (b) $\sum t_{2n+1} = \dfrac{12/27}{1-r}$ and $\sum t_{2n+2} = \dfrac{r \cdot 12/13}{1-r}$

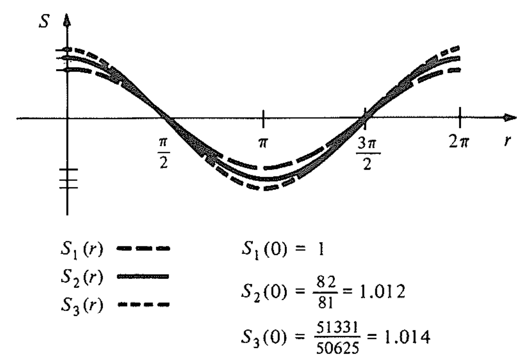
The sum is 1.

12.2 The Comparison Test and Alternating Series

1. Use $8/3^i$
3. Use $1/3^i$
5. Use $1/3^i$
7. Use $1/2^i$
9. Use $1/i$
11. Use $4/3i$
13. Converges
15. Converges
17. Diverges
19. Converges
21. Converges
23. Diverges
25. Converges
27. Diverges
29. Converges
31. Diverges
33. Converges
35. 0.29
37. 0.37
39. Diverges
41. Diverges
43. Converges absolutely
45. Diverges
47. Converges conditionally
49. Converges conditionally
51. -0.18
53. -0.087
55. Converges
57. (a) $a_1 = 2$, $a_2 = \sqrt{6}$, $a_3 = \sqrt{4+\sqrt{6}}$
(b) $\lim_{n\to\infty} a_n \approx 2.5616$
59. Increasing, bounded above. (Use induction.)
61. Increasing for $n \geqslant 2$, bounded above.
63. Show by induction that $a_2, a_3, \ldots$ is decreasing and bounded below, so converges. The limit l satisfies $l = \frac{1}{2}(l + \frac{B}{l})$.
65. $\lim_{n\to\infty} a_n = 4$
67. The limit exists by the decreasing sequence property.
69. Compare with $(3/4)^n$.

12.3 The Integral and Ratio Tests

1. Diverges
3. Converges
5. Converges
7. Converges
9. 0.44
11. Use Figure 12.3.2.
13. Converges
15. Converges
17. 11.54
19. (a) ≈ 1.708
(b) ≈ 1.7167
(c) 8 or more terms.
21. Converges
23. Diverges
25. Converges
27. Diverges
29. Converges
31. Converges
33. Converges
35. Converges
37. Show that if $|a_n|^{1/n} > 1$, then $|a_n| > 1$.
39. $p > 1$
41. $p > 1$
43. (a) $S - \frac{1}{2}f(n) = \sum_{i=1}^{n-1} f(i) + \frac{1}{2}f(n) + \frac{1}{2}\int_n^{n+1} f(x)\,dx + \int_{n+1}^{\infty} f(x)\,dx \leqslant \sum_{i=1}^{\infty} f(i) + \frac{1}{2}f(n) + \frac{1}{2}f(n) + \int_{n+1}^{\infty} f(x)\,dx$;
now use the hint.
(b) Sum the first 9 terms to get 1.0819. The first method saves the work of adding 6 additional terms.
45. (b)



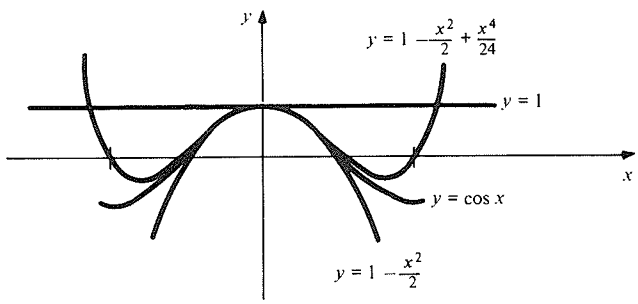
12.4 Power Series

1. Converges for $-1 \leqslant x < 1$.
3. Converges for $-1 \leqslant x \leqslant 1$.
5. Converges for $0 < x < 2$.
7. Converges for all x.
9. Converges for $-4 < x < 4$.
11. $R = \infty$
13. $R = 2$
15. $R = \infty$
17. $R = 1$, converges for $x = 1$ and -1.
19. $R = 3$
21. $R = 0$
23. Note that $f(0) = 0$ and $f'(0) = 1$.
25. (a) $R = 1$
(b) $\sum_{i=1}^{\infty} x^{i+1}$
(c) $f(x) = x(2-x)/(1-x)^2$ for $|x| < 1$
(d) 3
27. $\sum_{n=0}^{\infty}[(-1)^n x^{2n}/n!]$
29. $\tan^{-1}(x) = \sum_{n=0}^{\infty}[(-1)^n x^{2n+1}/(2n+1)]$, and $(d/dx)(\tan^{-1}x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.
31. $1/2 + 3x/4 + 7x^2/8 + 15x^3/16 + \cdots$
33. $x^2 - x^4/3 + 2x^6/45 + \cdots$
35. Set $f(x) = 1/(1-x)$ and $g(x) = -x^2/(1-x)$.
37. (a) $x + (1/3)x^3 + (2/15)x^5 + \cdots$
(b) $1 + x^2 + (2/3)x^4 + \cdots$
(c) $1 - x^2 + (1/3)x^4 - \cdots$
39. $\sum_{i=1}^{\infty}(-1)^{i+1}(1/i)x^i$
41. Use the fact that $\sqrt[i]{i} \to 1$ as $i \to \infty$.
43. Write $f(x) - f(x_0) = \left(f(x) - \sum_{i=0}^{N} a_i x^i\right) + \left(\sum_{i=0}^{N} a_i x^i - \sum_{i=0}^{N} a_i x_0^i\right) + \left(\sum_{i=0}^{N} a_i x_0^i - f(x_0)\right)$

45. Show that $f(x) = \int_0^x g(t)\,dt$.

12.5 Taylor's Formula

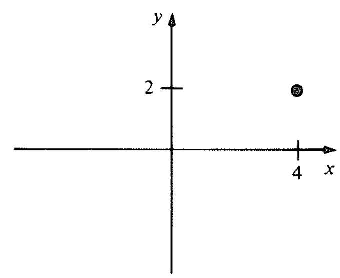
1. $3x - 9x^3/2 + 81x^5/40 - 243x^7/560 + \cdots$
3. $2 - 2x + 3x^2/2 - 4x^3/3 + 17x^4/24 - 4x^5/15 + 7x^6/80 - 8x^7/315 + \cdots$
5. $1/3 - 2(x-1)/3 + 5(x-1)^2/9 + 0\cdot(x-1)^3$
7. $e + e(x-1) + e(x-1)^2/2 + e(x-1)^3/6$.
9. (a) $1 - x^2 + x^6 + \cdots$ (b) 720
11. Valid if $-1 < x \leqslant 1$ (Integrate $1/(1+x) = 1 - x + x^2 - x^3 + \cdots$.)
13. Let $x - 1 = u$ and use the bionomial series.
15. (a) $1 - (1/2)x^2 + (3/8)x^4 - (5/16)x^6 + (35/128)x^8 - \cdots$
 (b) $(-1/2)(-1/2-1)\ldots(-1/2-10+1)\cdot(20!)/(10!)$
17. $f_0(x) = f_1(x) = 1$, $f_2(x) = f_3(x) = 1 - x^2/2$, $f_4(x) = 1 - x^2/2 + x^4/24$.



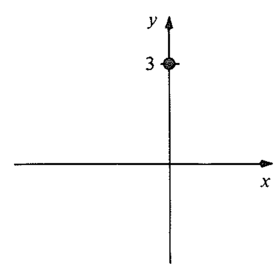
19. ≈ 0.095
21. ≈ 0.9
23. ≈ 0.401
25. (a) The remainder is less than $R^4M_3/12$ where M_3 is the maximum value of $|f'''(x)|$ on the interval $[x_0 - R, x_0 + R]$.
 (b) 0.9580. Simpsons rule gives 0.879.
27. $-4/3$
29. $1/6$
31. $\sum_{n=0}^{\infty} x^n$ for $|x| < 1$
33. $\sum_{n=0}^{\infty} 2x^{2n+1}$ for $|x| < 1$
35. $\sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$
37. $1 + 2x^2 + x^4$
39. $\int_1^x \ln t\,dt = \sum_{n=2}^{\infty}\{(-1)^i(x-1)^i/[i(i-1)]\}$.
 $x\ln x = (x-1) + \sum_{i=2}^{\infty}\{(-1)^i(x-1)^i/[i(i-1)]\}$.
 Conclude $\int_1^x \ln t\,dt = x\ln x - x + 1$.
41. $1, 0, 1/2, 0$
43. $0, -1, 0, -1/2$
45. $1/2 - x^2/4! + x^4/6! - \cdots$
47. $1 - 2x + 2x^2 - 2x^3 - 2x^4 + 2x^5 + \cdots$
49. (a) $(x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4$
 (b) $1 + (x-e)/e - (x-e)^2/2e^2 + (x-e)^3/3e^3 - (x-e)^4/4e^4$
 (c) $\ln 2 + (x-2)/2 - (x-2)^2/8 + (x-2)^3/24 - (x-2)^4/64$
51. $\ln 2 + x/2 + x^2/8 - x^4/192 + \cdots$
53. $\sin 1 + (\cos 1)x + [(\cos 1 - \sin 1)/2]x^2 - [(\sin 1)/2]x^3 + \cdots$
55. (a) 0.5869768
 (b) It is within $1/1000$ of $\sin 36°$.
 (c) $36° = \pi/5$ radians, and she used the first two terms of the Taylor expansion.
 (d) Use the fact that $10° = \pi/18$ radians and $\tan x \approx x(1 + x^2/3)$
57. (a) $0, -1/3, 0$
 (b) $1 - x^2/3! + x^4/5! - x^6/7! + \cdots$
59. Follow the method of Example 3(d).

12.6 Complex Numbers

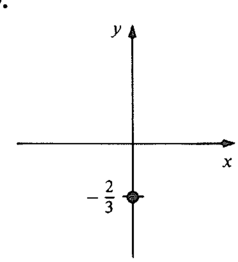
1. $-i$
3. $-i$
5.



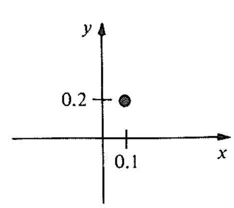
7.



9.

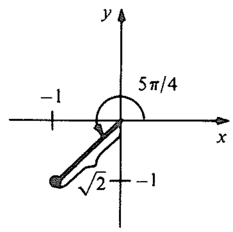


11.

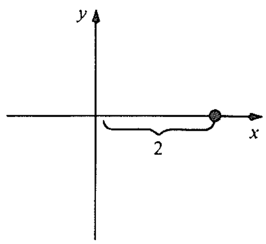


13. $-14 + 8i$
15. $3 + 4i$
17. $(5 + 3i)/34$
19. $(41 + 3i)/65$
21. $\pm\sqrt{3i}$
23. $(1 \pm \sqrt{17}\,i)/6$
25. $(7 \pm \sqrt{53})/2$
27. $\pm 2(1 + i)$
29. $\pm 2\sqrt{2}\,(i - 1)$
31. -1

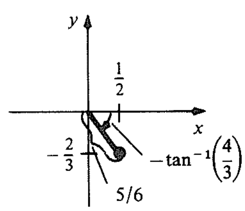
33. $-11/5$
35. $328/565$
37. $5-2i$
39. $\sqrt{3}-i/2$
41. $-1/3+2i/3$
43. $(-7+11i)/20$
45. 3
47. $|z|=\sqrt{2}$, $\theta=5\pi/4$



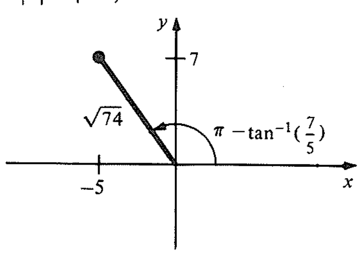
49. $|z|=2$, $\theta=0$



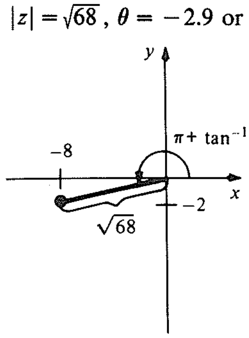
51. $|z|=5/6$, $\theta=-0.93$



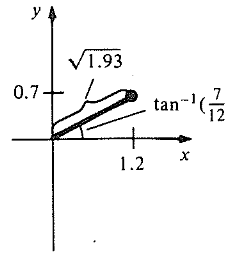
53. $|z|=\sqrt{74}$, $\theta=2.19$



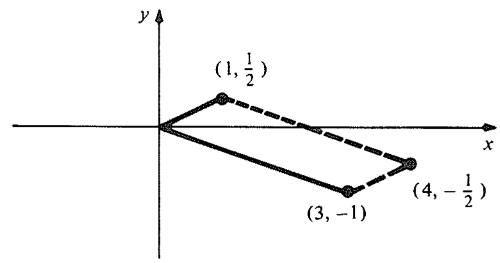
55. $|z|=\sqrt{68}$, $\theta=-2.9$ or 3.4



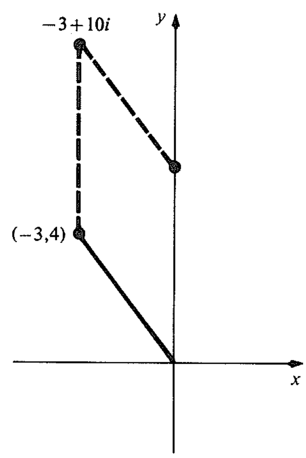
57. $|z|=\sqrt{1.93}$, $\theta=0.53$



59. Let $z_1=a+ib$, $z_2=c+id$ and calculate $|z_1z_2|$ and $|z_1|\cdot|z_2|$.
61. $(8+3i)^4$
63.

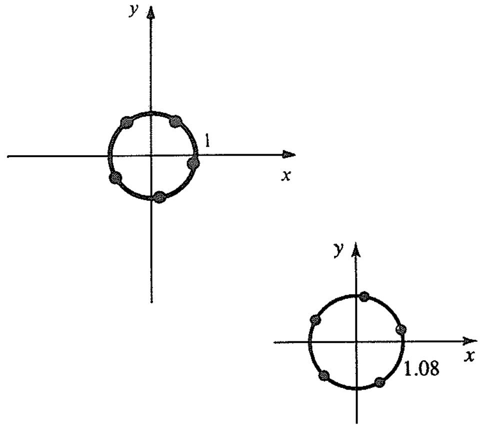


65.

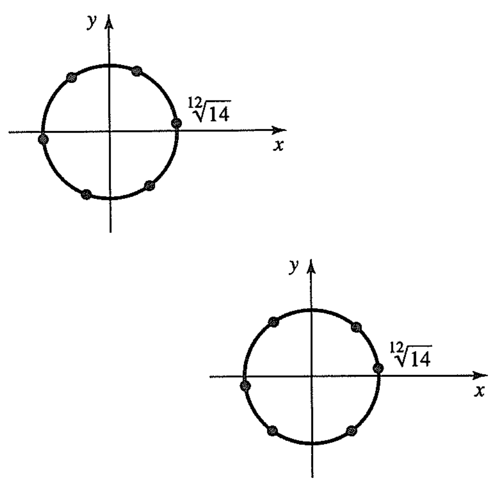


67. $2\sqrt{5}$
69. e^x, y
71. $1/2+\sqrt{3}\,i/2$
73. $-ei$
75. ei
77. $(3-4i)/25$
79. (a) $e^{ix}\cdot e^{-ix}=(\cos x+i\sin x)(\cos x-i\sin x)$; multiply out
(b) Show $e^z\cdot e^{-z}=1$ using (a).
81. Show $e^{3\pi i/2}=-i$.
83. Use $(e^{i\theta})^n=e^{in\theta}$.
85. $\sqrt{2}\,e^{i\pi/4}$
87. $(\sqrt{5}/5)e^{i(0.46)}$
89. $\sqrt{58}\,e^{i(-0.4)}$

91. $(\sqrt{37}/2)e^{i(-1.74)}$
93. $25e^{i(1.85)}$
95. $e^{i(\pi/15+2\pi k/5)}$, $k = 0, 1, 2, 3, 4$; $(1.08)\, e^{i(0.22+2\pi k/5)}$, $k = 0, 1, 2, 3, 4$



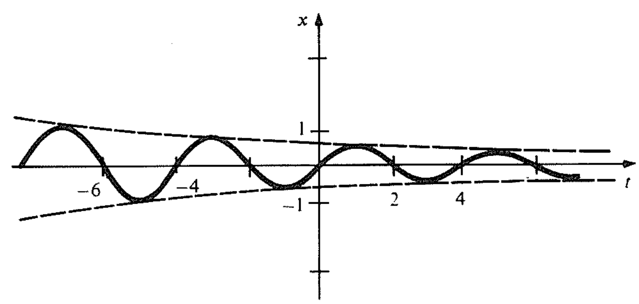
97. $\sqrt[12]{14}e^{i(0.155+2\pi k/6)}$, $k = 0, 1, 2, 3, 4, 5$; $\sqrt[12]{14}e^{i(0.107+2\pi k/6)}$, $k = 0, 1, 2, 3, 4, 5$



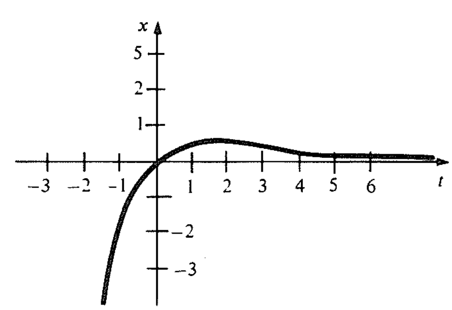
99. z is rotated by $\pi/4$ and its length multiplied by $1/\sqrt{2}$.
101. Show that $z^4 = 1$ and then that $z^2 = 1$.
103. Write $e^{i\theta} = \cos\theta + i\sin\theta$.
105. $$\frac{1}{2}\left(\sqrt{2}\,z + \frac{1}{\sqrt{\sqrt{2}-1}} - i\sqrt{\sqrt{2}-1}\right) \times \left(\sqrt{2}\,z - \frac{1}{\sqrt{\sqrt{2}-1}} + i\sqrt{\sqrt{2}-1}\right)$$
107. $(z + 2i + 2)(z - 2)$
109. (a) $\tan i\theta = i\tanh\theta$ (b) $\tan i\theta = (\tanh\theta)e^{i\pi/2}$
111. $z_1 = aiz_2$, a a real number
113. (a) Factor $z^n - 1$
(b) Use your factorization in (a).
(c) $-1, i, -i$
115. The motion of the moon with the sun at the origin.
117. (a) $(2n+1)\pi i$ for any integer n.
(b) You could define $\ln(-1) = i\pi$, although there are other possibilities.

12.7 Second-Order Linear Differential Equations

1. $y = c_1\exp(3x) + c_2\exp(x)$
3. $y = c_1\exp(x/3) + c_2\exp(x)$
5. $y = \frac{1}{2}\exp(3x) - \frac{1}{2}\exp(x)$
7. $y = e^x$
9. $y = c_1\exp[(2+i)x] + c_2\exp[(2-i)x] = \exp(2x)[a_1\cos x + a_2\sin x]$
11. $y = c_1\exp[(3+2i)x] + c_2\exp[(3-2i)x] = \exp(3x)[a_1\cos 2x + a_2\sin 2x]$
13. $y = x\exp(3x)$
15. $y = (x-1)\exp(-\sqrt{2} + \sqrt{2}\,x)$
17. (a) Underdamped
(b) $x = (1/\bar{\omega})(\sin\bar{\omega}t)\exp(-\pi t/32)$, $\bar{\omega} = \pi\sqrt{255}/32 \approx \pi/2$.



19. (a) Critically damped
(b) $x = t\exp(-\pi t/6)$



21. $y = c_1\exp(3x) + c_2\exp(x) + 2x + 6$.
23. $x = c_1\exp(t/3) + c_2\exp(t) + (2/5)\cos t + (-1/5)\sin t$
25. $y = e^{2x}(c_1\cos x + c_2\sin x) + x^2/5 + 13x/25 + 42/125$
27. $y = (c_1 + c_2x)\exp(\sqrt{2}\,x) + [(1+2\sqrt{2})/9]\cos x + [(1-\sqrt{2})/9]\sin x$

29. $y = c_1\exp(3x) + c_2\exp(x) + 2x + 6$

31. $x = c_1\exp(t/3) + c_2\exp(t) - \sin t/5 - 2\cos t/5$

33. $y = c_1\exp(3x) + c_2\exp(x) + [\exp(3x/2)]\int(\tan x)\exp(-3x)\,dx - [\exp(x)/2]\int(\tan x)\exp(-x)\,dx$

35. $y = e^{2x}(C_1\cos 2x + C_2\sin 2x) + [e^{2x}\cos 2x/2]\cdot\int\{e^{2x}[(1-\cot 2x)(\cos 2x) - (1+\cot 2x)(\sin 2x)]\cdot(1+\cos^2 x)\}^{-1}dx + \cdot\,[e^{2z}\sin 2x/2]\int\{e^{2x}[(1-\tan 2x)\cdot(\cos 2x) + (1+\tan 2x)(\sin 2x)](1+\cos^2 x)\}^{-1}dx$

37. $x = -\cos 2t + \cos t = 2\sin(3t/2)\sin(t/2)$

39. $x = (-1/24)\cos 5t + (1/5)\sin 5t + (1/24)\cos t$

41. (a)

$$x(t) = e^{-4t}\left[\frac{-40}{101\sqrt{21}}\sin(\sqrt{21}\,t) - \frac{42}{505}\cos(\sqrt{21}\,t)\right] + \frac{2}{\sqrt{505}}\cos\left[2t - \tan^{-1}\left(\frac{8}{21}\right)\right]$$

(b) Looks like $(2/\sqrt{505})\cos(2t - \tan^{-1}(8/21))$

43. (a) $x(t) = \exp(-t/2)[(7/10)\cos(\sqrt{15}\,t/2) + (-1/2\sqrt{15})\cdot\sin(\sqrt{15}\,t/2)] + (1/\sqrt{10})\cos(t - \tan^{-1}(1/3))$

(b) Looks like $(1/\sqrt{10})\cos(t - \tan^{-1}(1/3))$.

45. Show that the Wronskian of y_1 and y_2 does not vanish.

47. (a) Subtract two solutions with the same initial conditions.
(b) Show that they are zero when $x = 0$.
(c) Solve algebraically for $y(x)$.

49. (a) Compute the derivative of the Wronskian
(b) If $(\alpha-1)^2 \neq 4\beta$ and r_1, r_2 are roots, then $y = c_1x^{r_1} + c_2x^{r_2}$; if $(\alpha-1)^2 = 4\beta$ and r is the root, then $y = c_1x^r + c_2x^{(1-\alpha)/2}\ln x$. (Assume $x > 0$ in each case).

51. (a) Add all three forces
(b) Substitute and differentiate.

53. $c_1e^{\lambda} + c_2e^{i\lambda} + c_3e^{-\lambda} + c_4e^{-i\lambda}$ where $\lambda = (1+i)/\sqrt{2}$ or

$$e^{x/\sqrt{2}}\left[b_1\cos\left(\frac{x}{\sqrt{2}}\right) + b_2\sin\left(\frac{x}{\sqrt{2}}\right)\right] + e^{-x/\sqrt{2}}\left[b_3\cos\left(\frac{x}{\sqrt{2}}\right) + b_4\sin\left(\frac{x}{\sqrt{2}}\right)\right]$$

55. $\frac{1}{2}e^x + f(x)$, where $f(x)$ is the solution to Exercise 53.

12.8 Series Solutions of Differential Equations

1. $y = a_0\left[\sum_{1=0}^{\infty}\frac{x^{2n}}{2^n n!}\right] + a_1\left[\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}\right]$

3. $y = a_0 + a_1x + a_1\left[\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(n+1)!}\right]$

5. $y = x - x^3/3 + x^5/10 - \cdots$

7. $y = 2x - x^3 + 7x^5/20 - \cdots$

9. $y = a_0\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + a_1\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right)$

The recursion relation is

$$a_{n+3} = a_n/[(n+3)(n+2)]$$

11. $y = c_1\left(1 + x - \frac{x^2}{4} + \frac{x^3}{60} - \frac{x^4}{1920} + \cdots\right) + c_2\left(x^{4/3} - \frac{x^{7/3}}{7} + \frac{x^{10/3}}{140} - \frac{x^{13/3}}{5460} + \cdots\right)$

13. $c_1x^{(1+11i)/6} + c_2x^{(1-11i)/6}$, or $x^{1/6}\left[b_1\cos\left(\frac{11\ln x}{6}\right) + b_2\sin\left(\frac{11\ln x}{6}\right)\right]$ (no further terms).

15. (a) $x^k + \frac{x^{k+2}}{4k+4} + \frac{x^{k+4}}{(4k+4)(8k+16)} + \cdots + \frac{x^{k+2j}}{4^j(k+1)(2k+4)\cdots(jk+j^2)} + \cdots$

(b) $x^{-k} + \frac{x^{-k+2}}{-4k+4} + \frac{x^{-k+4}}{(-4k+4)(-8k+16)} + \cdots + \frac{x^{-k+2j}}{4^j(-k+1)(-2k+4)\cdots(-jk+j^2)} + \cdots$

17. Solve recursively for coefficients, then recognize the series for sine and cosine.

19. (a) Use the ratio test
(b) $x,\ -\frac{1}{2} + \frac{3}{2}x^2,\ -\frac{3}{2}x + \frac{5}{2}x^3$

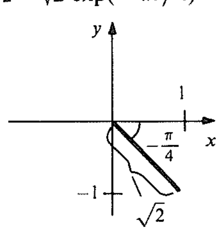
21. Show that the Wronskian is non-zero

23. (a) Solve recursively
(b) Substitute the given function in the equation. (To discover the solution, use the methods for solving first order linear equations given in Section 8.6).

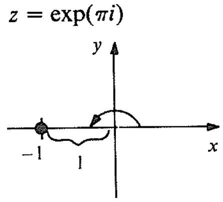
Review Exercises for Chapter 12

1. Converges to $1/11$.
3. Converges to $45/2$
5. Converges to $7/2$
7. Diverges
9. Converges
11. Converges
13. Converges
15. Diverges
17. Diverges
19. Converges
21. Converges

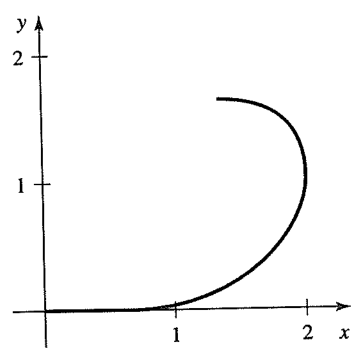
23. Converges
25. 0.78
27. -0.12
29. -0.24
31. 0.25
33. False
35. False
37. False
39. False
41. True
43. True
45. True
47. Use the comparison test.
49. $1/8$
51. 1
53. $R = \infty$
55. $R = \infty$
57. $R = 2$
59. $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_{2i} = \dfrac{2^{2i}}{(2i)!}$, $a_{2i+1} = \dfrac{(-1)^i 3^{2i+1} + 2^{2i+1}}{(2i+i)!}$
61. $\sum_{i=1}^{\infty}[(-1)^{i+1}x^{4i}/i]$
63. $\sum_{i=1}^{\infty}[(-1)^i x^{2i}/(2i)!]$
65. $\sum_{i=1}^{\infty}[x^i/[i(i!)]]$
67. $\sum_{i=0}^{\infty}[e^2(x-2)^i/i!]$, $R = \infty$
69. $\displaystyle\sum_{i=0}^{\infty} \frac{\frac{3}{2}(\frac{3}{2}-1)\cdots(\frac{3}{2}-i+1)}{i!}(x-1)^i$
71. $\pi^2/2$
73. 3
75. $3, 7, 3-7i, \sqrt{58}$
77. $\pm(1+\frac{1}{2}\sqrt{5})^{1/2} \approx \pm 1.46$, $\mp(-1+\frac{1}{2}\sqrt{5})^{1/2} \approx \mp 0.344$, $\sqrt{2+i} \approx \pm 1.46 \pm 0.344i$, $\sqrt[4]{5} \approx 1.50$
79. $z = \sqrt{2}\exp(-\pi i/4)$



81. $z = \exp(\pi i)$



83. $1 \pm \sqrt{1-\pi i} \approx 2.4658 - 1.0717i$, $-0.4658 + 1.0717i$
85. $c_1\cos 2x + c_2\sin 2x$
87. $y = c_1\exp(-5x) + c_2\exp(-x)$
89. $y = -e^x/6 - 11\cos x/130 + 33\sin x/130 + c_1\exp(-5x) + c_2\exp(2x)$
91. $c_1e^{3x} + c_2xe^{3x} + \dfrac{140}{1369}\cos\left(\dfrac{x}{2}\right) - \dfrac{48}{1369}\sin\left(\dfrac{x}{2}\right)$
93. $\displaystyle -\cos(2x)\int\frac{x\sin(2x)}{\sqrt{x^2+1}}\,dx + \sin(2x)\int\frac{x\cos(2x)}{\sqrt{x^2+1}}\,dx$
95. $c_1 + e^{-x}(c_2\cos x + c_3\sin x)$
97. $m = 1$, $k = 9$, $\gamma = 1$, $F_0 = 1$, $\Omega = 2$, $\omega = 3$, $\delta = \tan^{-1}(\frac{1}{2})$. As $t \to \infty$, the solution approaches $\frac{1}{\sqrt{34}}\cos[2t - \tan^{-1}(\frac{2}{5})]$.
99. $m = 1$, $k = 25$, $\gamma = 6$, $F_{\_} = 1$, $\Omega = \pi$, $\omega = 5$, $\delta = \tan^{-1}[6\pi/(25-\pi^2)]$. As $t \to \infty$, the solution approaches
$$\frac{1}{\sqrt{625 - 14\pi^2 - \pi^4}}\cos\left[\pi t - \tan^{-1}\left(\frac{6\pi}{25-\pi^2}\right)\right]$$
101. $a_0\left(1 - \dfrac{x^3}{3} + \cdots\right) + a_1\left(x - \dfrac{x^4}{6} + \cdots\right)$
103. $a_0\left(1 - x^2 + \dfrac{x^4}{6} - \dfrac{x^5}{5} + \cdots\right) + a_1\left(x - \dfrac{x^3}{3} + \dfrac{x^4}{6} - \dfrac{x^5}{20} + \cdots\right)$
105. $1 - x + \dfrac{x^2}{2} - \dfrac{11x^3}{6} + \cdots$
107. $x + \dfrac{x^3}{8} + \dfrac{x^5}{192} + \cdots$
109. (a) $m = L$, $k = 1/C$, $\gamma = R$
(b) $0.01998e^{-19.90t} - 0.02020e^{-0.1005t} + 0.002099\sin(60\pi t) + 0.0002227\cos(60\pi t)$
111. Factor out x^2.
113. (a) The partial sums converge to $y(x,t)$ for each (x,t).
(b) $\sum_{k=0}^{\infty}(-1)^k A_{2k+1}$
115. ≈ 0.659178
117. (a) ≈ 1.12
(b) ≈ 2.24. It is accurate to within 0.02.
119. $-1/2, 1/6, 0$
121. (a) ≈ 3.68
(b)



123. Show by induction that $g^{(n)}(x)$ is a polynomial times $g(x)$.
125. True
127. (a) Collect terms
(b) The radius of convergence is at most 1.
(c) e
129. Show that $\alpha < 1/k$ by using a Maclaurin series with remainder.